\textbf{\textit{C}}*-\textit{algebras arising from integral and rational dynamics}

Tron "James" Omland\textsuperscript{1}

(based on joint work with Selçuk Barlak\textsuperscript{2} and Nicolai Stammeier\textsuperscript{1})

\textsuperscript{1}University of Oslo, \textsuperscript{2}University of Southern Denmark

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Cuntz (2008) introduces $\mathcal{Q}_N$ as the universal $C^*$-algebra generated by isometries $\{s_n\}_{n \in \mathbb{N}}$ and a unitary $u$ satisfying

$$s_m s_n = s_{mn}, \quad s_n u = u^n s_n, \quad \text{and} \quad \sum_{k=0}^{n-1} u^k s_n s_n^* u^{-k} = 1.$$ 

Larsen and Li (2012) define $\mathcal{Q}_2$ as the universal $C^*$-algebra generated by an isometry $s_2$ and a unitary $u$ satisfying

$$s_2 u = u^2 s_2 \quad \text{and} \quad s_2 s_2^* + u s_2 s_2^* u^* = 1.$$ 

We think of $\mathcal{Q}_N$ as coming from the set $S = \{\text{all primes}\}$, and $\mathcal{Q}_2$ as coming from $S = \{2\}$. It is computed that

$$K_0(\mathcal{Q}_N) \cong \mathbb{Z}^\infty \cong K_1(\mathcal{Q}_N) \quad \text{and} \quad K_0(\mathcal{Q}_2) \cong \mathbb{Z} \cong K_1(\mathcal{Q}_2).$$

The more general versions can have torsion in their $K$-groups.
The algebras $Q_S$

**Definition**

Let $S$ be a set of mutually relatively prime numbers $\geq 2$. Define the algebra $Q_S$ as the universal $C^*$-algebra generated by a unitary $u$ and isometries $\{s_p\}_{p \in S}$ satisfying

(i) $s_p^* s_q = s_q s_p^*$,

(ii) $s_p u = u^p s_p$, and

(iii) $\sum_{k=0}^{p-1} e_{k+p\mathbb{Z}} = 1$

for all $p, q \in S$, where $e_{k+p\mathbb{Z}} := u^k s_p s_p^* u^{-k}$.

Let $(\xi_n)_{n \in \mathbb{Z}}$ denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$. If we define

$$U \xi_n = \xi_{n+1} \quad \text{and} \quad S_p \xi_n = \xi_{pn},$$

then $U$ and $\{S_p\}_{p \in S}$ satisfy (i)–(iii). This representation on $\ell^2(\mathbb{Z})$ is faithful, so we can also think of $Q_S$ as a subalgebra of $B(\ell^2(\mathbb{Z}))$. 
Let $H^+$ be the submonoid of $\mathbb{N}^\times$ generated by $S$. Then

$$Q_S \cong (\mathcal{D}_S \times \mathbb{Z}) \rtimes^e H^+ \cong \mathcal{D}_S \rtimes^e (\mathbb{Z} \rtimes H^+),$$

where

$$\mathcal{D}_S = C^* \{e_{k+q\mathbb{Z}} : q \in S, k \in \mathbb{Z}\}.$$

Set $N = \mathbb{Z}[\{\frac{1}{p} : p \in S\}] \subseteq \mathbb{Q}$ and let $H$ be the subgroup of $\mathbb{Q}_+^\times$ generated by $S$. Then it follows from the dilation theory of Laca that

$$Q_S \sim_M C_0(\Omega) \rtimes N \rtimes H,$$

where $\Omega$ is the completion of $N$ w.r.t. to the subgroup topology generated by $\{h\mathbb{Z} : h \in H\}$, and the action is the natural $ax + b$-action.

Let $\Delta$ be the closure of $\mathbb{Z}$ in $\Omega$, then $\mathcal{D}_S \cong C(\Delta)$.

Moreover, $Q_S$ is isomorphic to the full corner of $C_0(\Omega) \rtimes N \rtimes H$ cut down by the projection $\chi_\Delta \in C_0(\Omega)$.

If $P = \{p \in \mathbb{N}^\times : p$ prime and $p|q$ for some $q \in S\}$, then $\Delta \cong \prod_{p \in P} \mathbb{Z}_p$ and $\Omega \cong \prod_{p \in P} Q_p$. 
\( Q_S \) can also be constructed from either \( \mathbb{N} \rtimes H^+ \) or \( \mathbb{Z} \rtimes H^+ \) using the theory of boundary quotients of semigroup \( C^\ast \)-algebras.

Relatively primeness of \( S \) gives that \( \mathbb{N} \rtimes H^+ \) and \( \mathbb{Z} \rtimes H^+ \) are right LCM. Both are also left Ore semigroups with enveloping group \( N \rtimes H \subseteq \mathbb{Q} \rtimes \mathbb{Q}_+^\times \), where still \( N = \mathbb{Z}[\{\frac{1}{p} : p \in S\}] \).

First, note that \((\mathbb{N} \rtimes H^+, N \rtimes H)\) forms a quasi-lattice ordered group. Hence we can form the Toeplitz algebra \( \mathcal{T}(\mathbb{N} \rtimes H^+, N \rtimes H) \) using the work of Nica, which coincides with \( C^\ast(\mathbb{N} \rtimes H^+) \).

To define \( C^\ast(\mathbb{Z} \rtimes H^+) \), we use Xin Li’s theory of semigroup \( C^\ast \)-algebras, which generalizes Nica’s approach.

Boundary quotients were introduced by Crisp and Laca for quasi-lattice ordered groups, and later generalized to right LCM semigroups by several people. In this setting

\[
BQ(\mathbb{N} \rtimes H^+) = BQ(\mathbb{Z} \rtimes H^+) = Q_S.
\]
Theorem

For every set $S$ of relatively prime numbers, the algebra $\mathcal{Q}_S$ is a unital Kirchberg algebra in the UCT class. Consequently, the $K$-theory is a complete isomorphism invariant for $\mathcal{Q}_S$ (Kirchberg-Phillips).

We can use that $\mathcal{Q}_S$ is a full corner of $C_0(\Omega) \rtimes N \rtimes H$ to see this: The $ax + b$-action of $N \rtimes H$ on $\Omega$ is minimal, locally contractive, and topologically free, implying that $C_0(\Omega) \rtimes N \rtimes H$ is purely infinite and simple (Archbold-Spielberg, Laca-Spielberg).

Separability, nuclearity, UCT hold because:
$N \rtimes H$ is discrete countable amenable, $C_0(\Omega)$ is commutative separable, and the transformation groupoid of $(\Omega, N \rtimes H)$ is amenable (Tu).
Let $S$ be a set of mutually relatively prime numbers. Define

$$g = \gcd\{p - 1 : p \in S\} = \max\{q \in \mathbb{N}^\times : q|p - 1 \text{ for all } p \in S\}.$$  

If $2 \in S$, then $g = 1$.

**Theorem**

\[ K_i(\mathcal{Q}_S) \cong \mathbb{Z}^{2^{|S|} - 1} \oplus T_i, \quad i = 0, 1, \]

where $T_0$ and $T_1$ are torsion groups, which are finite if $S$ is finite. Moreover, if $g = 1$, then $T_0$ and $T_1$ are both trivial.

Case $|S| = 1$, i.e., $S = \{r\}$ is previously studied (Hirshberg, Katsura). These are graph $C^*$-algebras and their $K$-theory is given by

\[ K_0(\mathcal{Q}_S) \cong \mathbb{Z} \oplus \mathbb{Z}/(r - 1), \quad [1]_0 = (0, 1), \quad K_1(\mathcal{Q}_S) = \mathbb{Z}. \]

Hence, $\mathcal{Q}_{\{r\}} \cong \mathcal{Q}_{\{q\}}$ if and only if $r = q$. 
Theorem

Assume $|S| = 2$, i.e., $S = \{q, r\}$ and $g = \gcd\{q - 1, r - 1\}$. Then

$$K_0(Q_S) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}, \quad [1]_0 = (0, 1), \quad K_1(Q_S) = \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}.$$

E.g. $Q_{\{4,13\}} \cong Q_{\{7,10\}}$.

When $|S| \geq 3$ and $g > 1$, we can only show that $|T_i|$ divides $g^{2|S| - 2}$.

Conjecture

For $|S| \geq 2$ we have

$$T_0 \cong (\mathbb{Z}/g\mathbb{Z})^{2|S| - 2} \cong T_1,$$

and consequently, $Q_S \cong Q_{S'}$ if and only if $|S| = |S'|$ and $g = g'$. 

Tron Omland, UiO
The algebras $Q_H$

**Definition**

Let $H \subseteq \mathbb{Q}^\times$. Define the algebra $Q_H$ as the universal $C^*$-algebra generated by a unitary $u$ and partial isometries $\{s_h\}_{h \in H}$ satisfying

1. $s_h^* = s_{h^{-1}}$ and $s_h^* s_h s_g = s_h^* s_{hg}$ for all $g, h \in H$.
2. $s_h u^q = u^p s_h$ when $h = p/q$.
3. $\sum_{k=0}^{p-1} e_{k+p\mathbb{Z}} = 1$, where $e_{k+p\mathbb{Z}} := u^k s_h s_h^* u^{-k}$ and $h = p/q$.

Let $(\xi_n)_{n \in \mathbb{Z}}$ denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$. If we define

$$U \xi_n = \xi_{n+1} \quad \text{and} \quad S_h \xi_n = \xi_{hn} \text{ if } hn \in \mathbb{Z} \text{ and } 0 \text{ else},$$

then $U$ and $\{S_h\}_{h \in H}$ satisfy (i)–(iii). This representation on $\ell^2(\mathbb{Z})$ is faithful, so we can also think of $Q_H$ as a subalgebra of $B(\ell^2(\mathbb{Z}))$. Moreover, if $h = p/q$ with $\text{gcd}(p, q) = 1$, then $S_h = S_p S_q^*$ so that $S_h^* S_h = E_{q\mathbb{Z}}$ and $S_h S_h^* = E_{p\mathbb{Z}}$. 

Set $N = \mathbb{Z}[\{ h : h \in H \}] \subseteq \mathbb{Q}$ and let $\Omega$ be the completion of $N$ w.r.t. to the subgroup topology generated by $\{ h\mathbb{Z} : h \in H \}$, i.e., $N$ is dense in $\Omega$. Let $\Delta$ be the closure of $\mathbb{Z}$ in $\Omega$.

**Proposition**

The algebra $Q_H$ embeds into $C_0(\Omega) \rtimes N \rtimes H$ as a full corner, cut down by the projection $\chi_\Delta \in C_0(\Omega)$. Thus, when $H$ is infinite, $Q_H$ is a UCT Kirchberg algebra, and its $K$-theory is a complete isomorphism invariant (Kirchberg-Phillips).
Partial actions

**Definition**

A partial action of a group $G$ on a set $X$ is a collection $\{D_g\}_{g \in G}$ of subsets of $X$, and a collection of maps $\{\theta_g\}_{g \in G}$, $\theta_g : D_{g^{-1}} \rightarrow D_g$ such that

$$D_e = X, \quad \theta_e = \text{id}_X$$

$\theta_{gh}$ is an extension of $\theta_g \circ \theta_h$

**Example**

Every $H \subseteq \mathbb{Q}^\times$ acts partially on $\mathbb{Z}$ as follows: For $h = p/q \in H$ with $\gcd(p, q) = 1$, set $D_h = p\mathbb{Z}$, and define $\theta_h : q\mathbb{Z} \rightarrow p\mathbb{Z}$ by $qn \rightarrow pn$. 
Let $\mathcal{B}_H$ be the $C^*$-subalgebra of $\mathcal{Q}_H$ generated by $u$ and projections $\{e_{k+q\mathbb{Z}} : k \in \mathbb{Z}, 1/q \in \mathbb{N}\}$. Recall that $\mathcal{B}_H \cong \mathcal{D}_H \rtimes \mathbb{Z}$ is a Bunce-Deddens algebra.

The group $H$ acts partially by $\alpha$ on $\mathcal{B}_H$, where each $\alpha_h$ is a $\ast$-isomorphism from its domain $D_{h^{-1}} = e_{q\mathbb{Z}}\mathcal{B}_He_{q\mathbb{Z}}$ to its range $D_h = e_{p\mathbb{Z}}\mathcal{B}_He_{p\mathbb{Z}}$. In terms of the generators $u$ and $e_X$, the map $\alpha_h$ for $h = p/q$ with $\gcd(p, q) = 1$ is defined by

(i) for $X \subseteq q\mathbb{Z}$ by $\alpha_h(e_X) := e_{hX}$, and

(ii) for $n \in q\mathbb{Z}$ by $\alpha_h(u^n) = u^{hn}$.

Remark

In Exel’s definition of a partial $C^*$-dynamical systems, the domains are required to be ideals, which is not the case here. We would still like to think about our $\mathcal{Q}_H$ as a partial crossed product $\mathcal{B}_H \rtimes_{\alpha}^{\text{part}} H$. 
Let $H \subseteq \mathbb{Q}_+^\times$ be nontrivial and choose a minimal generating set $\{p_i/q_i\}_{i \in I}$ such that $\gcd(p_i, q_i) = 1$ and $p_i > q_i$ for all $i \in I$. Define

$$g = \gcd\{p_i - q_i : i \in I\} = \max\{r \in \mathbb{N}^\times : r|(p_i - q_i) \text{ for all } i \in I\}.$$
Two-generator case and conjecture

Remark (one-generator case)

Let $H = \langle p/q \rangle$, with $\gcd(p, q) = 1$ and $p > q$.
Then $T_0 = \mathbb{Z}/(p - q)\mathbb{Z}$ and $T_1 = 0$.

Theorem (two-generator case)

Assume $H = \langle p_1/q_1, p_2/q_2 \rangle \cong \mathbb{Z}^2$, and set $g = \gcd\{p_1 - q_1, p_2 - q_2\}$.
Then

$$K_0(\mathcal{Q}_H) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}, \quad [1]_0 = (0, 1), \quad K_1(\mathcal{Q}_H) = \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}.$$  

Conjecture

When the rank $m$ of $H$ is at least 2, we have

$$K_i(\mathcal{A}_H) = T_i \cong (\mathbb{Z}/g\mathbb{Z})^{2^{m-2}} \quad \text{for } i = 1, 2,$$

and consequently, $\mathcal{Q}_H \cong \mathcal{Q}_{H'}$ if and only if $m = m'$ and $g = g'$. 
Techniques involved in the proof

Step 1 (comparing with real dynamics)

Again, let $H$ be the subgroup of $\mathbb{Q}^\times$.
Recall that $N = \mathbb{Z}[\{h : h \in H\}]$ and $\Omega$ is a completion of $N$, and that $\mathcal{Q}_H \sim_M C_0(\Omega) \rtimes N \rtimes H$.
Then we apply a “duality theorem” (Kaliszewski-O.-Quigg, 14):

$$C_0(\Omega) \rtimes_{ax+b} (N \rtimes H) \sim_M C_0(\mathbb{R}) \rtimes_{ax+b} (N \rtimes H)$$

Hence, the problem is to compute the $K$-theory of $C_0(\mathbb{R}) \rtimes N \rtimes H$. 
Step 2 (decomposition)

The embedding $C_0(\mathbb{R}) \rtimes H \hookrightarrow C_0(\mathbb{R}) \rtimes N \rtimes H$ induces an injection in $K$-theory onto the free abelian part of $K_*(C_0(\mathbb{R}) \rtimes N \rtimes H) = K_*(\mathbb{Q}_H)$. The action of $H$ is homotopic to the trivial action, so

$$K_*(C_0(\mathbb{R}) \rtimes H) = K_*(C_0(\mathbb{R}) \otimes C^*(H)) = K_*(C_0(\mathbb{R}) \otimes C^*(\mathbb{Z}^m)) = \mathbb{Z}^{2^m-1}$$

There is a certain $H$-invariant subalgebra $A \subset C_0(\mathbb{R}) \rtimes N$ such that

$$K_*(C_0(\mathbb{R}) \rtimes N \rtimes H) \cong K_*(C_0(\mathbb{R}) \rtimes H) \oplus K_*(A \rtimes H),$$

where $K_*(A \rtimes H)$ is a torsion group.

(for this, one first shows that $K_*(C_0(\mathbb{R}) \rtimes N) \cong K_*(C_0(\mathbb{R})) \oplus K_*(A)$)
Step 3 (the torsion part)
The algebra $A_H$ can be described as follows:
Consider $M_H \subset B_H \cong C(\Delta) \rtimes \mathbb{Z} = \chi_\Delta(C_0(\Omega) \rtimes N)\chi_\Delta$.

There is an $H$-invariant $C^*$-subalgebra $B$ of $C_0(\Omega) \rtimes N$, such that $M_H$ is a full corner of $B$ cut down by $\chi_\Delta$.

Since $Q_H$ embeds as a full corner of $C_0(\Omega) \rtimes N \rtimes H$ cut down by $\chi_\Delta$, we can find a $C^*$-subalgebra $A_H$ of $Q_H$ that embeds as a full corner of $B \rtimes H$ cut down by $\chi_\Delta$.

There exists an $H$-equivariant isomorphism $C_0(\Omega) \rtimes N \cong C_0(\mathbb{R}) \rtimes N$, and then the $A$ above is defined as the image of $B$ under this map.

The partial action of $H$ on $B_H$ restricts to a partial action on $M_H \subset B_H$, where the domains become $D_h = e_q\mathbb{Z}M_He_q\mathbb{Z}$. One might then think of $A_H$ as $M_H \rtimes_\alpha^{\text{part}} H$. 
Other descriptions of $\mathcal{A}_S$

In general, it remains to find good descriptions of $\mathcal{A}_H$, but in the original case where $H$ is generated by a set $S$ of mutually relatively prime numbers, we have that

$$\mathcal{A}_S = C^* \{ u^m s_p \mid p \in S, 0 \leq m \leq p - 1 \} \subset \mathcal{Q}_S.$$ 

Moreover, recall that

$$\mathcal{Q}_S \cong (\mathcal{D}_S \rtimes \mathbb{Z}) \rtimes^e H^+,$$

and the UHF-algebra $M_{d\infty}$, for $d = \prod_{p \in S} p$, is a subalgebra of $\mathcal{D}_S \rtimes \mathbb{Z}$ invariant under $H^+$, so

$$\mathcal{A}_S \cong M_{d\infty} \rtimes^e H^+,$$

and we can show that $\mathcal{A}_S$ is a UCT Kirchberg algebra. For $|S| \geq 2$, both $K$-groups of $\bigotimes_{p \in S} \mathcal{O}_p$ are $(\mathbb{Z}/g\mathbb{Z})^{2|S|-2}$. Hence, our conjecture about $K_*(\mathcal{Q}_S)$ is equivalent with the following:

**Conjecture (restated for $\mathcal{Q}_S$)**

The algebra $\mathcal{A}_S$ is isomorphic to $\bigotimes_{p \in S} \mathcal{O}_p$. 