Metric Diophantine approximation: solubility of inhomogeneous wave equation

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Diophantine Approximation

How well can a real number be approximated by rationals?

- **Qualitatively**: The set of rational numbers is dense, therefore for any real number \( x \) we can construct a sequence of rational numbers \( r_n \) such that \( r_n \to x \) as \( n \to \infty \).

- **Quantitatively**: What happens if the denominators of rational numbers are equal to some integer value \( N \)?

\[
\left| x - \frac{p}{q} \right| \leq \frac{1}{2N}
\]

- What happens if the denominators are bounded by some value \( N \)?
Theorem (Dirichlet 1842)

For any real number \( \alpha \) and any positive integer \( N \), there exists a rational \( p/q \) with positive denominator \( q \leq N \), such that

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{qN}.
\]

Since \( q \leq N \), it immediately follows that

Corollary

For any irrational \( \alpha \in \mathbb{R} \) there exists infinitely many rationals \( p/q \) such that

\[
\left| \alpha - \frac{p}{q} \right| < q^{-2}.
\]
Is it possible to do better?

Theorem (Hurwitz 1891)

For any irrational real number \( \alpha \) there exist infinitely many integers \( p \) and \( q > 0 \) such that

\[
|\alpha - \frac{p}{q}| \leq \frac{1}{\sqrt{5}q^2}.
\]

For \( \alpha = \frac{\sqrt{5} + 1}{2}, \sqrt{5} \) in the above inequality is best possible.

A real number \( \alpha \) is said to be **badly approximable** if there exists a constant \( c = c(\alpha) > 0 \) such that

\[
|\alpha - \frac{p}{q}| > \frac{c}{q^2}, \quad \text{for all integers } p \text{ and } q > 0
\]

All quadratic irrationals are badly approximable

\[
W(\tau) := \{\alpha \in \mathbb{R} : |\alpha - \frac{p}{q}| \leq q^{-\tau} \text{ for i.m.}(p, q) \in \mathbb{Z} \times \mathbb{N}\}.
\]

How big are the sets \( W(\tau) \) for \( \tau > 2 \), **Bad** and \( \mathcal{L} \)?
An \textit{approximating function} is a function \( \psi : \mathbb{N} \to \mathbb{R}^+ \) such that \( \psi(r) \to 0 \) as \( r \to \infty \).

\( W(\psi) := \{ \alpha \in \mathbb{I} : |\alpha - p/q| < \psi(q) \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N} \} \).

\textbf{Theorem (Khintchine(24, 25))}

Let \( \psi \) be an approximating function. Then

\[
|W(\psi)| = \begin{cases} 
0 & \text{if } \sum_{q=1}^{\infty} q\psi(q) < \infty \\
1 & \text{if } \sum_{q=1}^{\infty} q\psi(q) = \infty \quad \psi \text{ is decreasing.}
\end{cases}
\]
**Duffin-Schaeffer Conjecture (1941).** For any function $\psi : \mathbb{N} \mapsto \mathbb{R}^+$

$$|W(\psi) \cap \mathbb{I}| = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \varphi(q) \psi(q) = \infty.$$

**Littlewood’s Conjecture (1930).** For any real $\alpha$ and $\beta$ and any $\epsilon > 0$ there are infinitely many positive integers $q$ such that

$$q \|q\alpha\| \|q\beta\| < \epsilon.$$

**Theorem (Einsiedler–Katok–Lindenstrauss 2006)**

The set of exceptions to the Littlewood’s conjecture has zero Hausdorff dimension.
Open Problems

**Baker–Schmidt Problem (1970)**

*For any non-degenerate submanifold $\mathcal{M}$ of $\mathbb{R}^n$,*

$$\dim(W(\tau) \cap \mathcal{M}) = \frac{n + 1}{\tau + 1} + \dim \mathcal{M} - 1$$

- **Kleinbock–Margulis (1996).** Almost all points on $\mathcal{M}$ are not very well approximable ($\mathcal{M}$ is extremal).

**Generalised Baker–Schmidt Problem (GBSP)**

*Determine Hausdorff measure for $W(\psi) \cap \mathcal{M}$, especially the convergent case.*

- **Hussain (2015).** GBSP holds for a parabola.
- **Huang, J. J.** (To appear, CRELLE). GBSP holds for any non-degenerate planar curve.
- **Badziahin–Harrap–Hussain (preprint).** Inhomogeneous GBSP holds for any non-degenerate planar curve.
A Diophantine Problem

Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be periodic in each of its variables \( x_1, x_2 \) and \( t \), with periods \( \alpha, \beta \) and \( \gamma \) respectively. Assume also that \( f \) is a smooth function of each variable. The inhomogeneous wave equation is given by the PDE

\[
\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x_1^2} - \frac{\partial^2 u(x, t)}{\partial x_2^2} = f(x, t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \ t \in \mathbb{R}, \quad (2)
\]

where \( u \) is a smooth, periodic solution with the same periods as \( f \). The smoothness conditions on \( f \) are equivalent to the property that it has a Fourier series expansion of the form

\[
f(x, t) = \sum_{(a,b,c) \in \mathbb{Z}^3} f_{a,b,c} \exp \left( 2\pi i \left[ \frac{a}{\alpha} x_1 + \frac{b}{\beta} x_2 + \frac{c}{\gamma} t \right] \right),
\]

in which the coefficients \( f_{a,b,c} \) decay faster than the reciprocal of any polynomial in \( (a, b, c) \) as \( \max\{|a|, |b|, |c|\} \) tends to infinity. Any smooth solution \( u \) to (2) must satisfy a similar Fourier expansion.
A Diophantine Problem

That is \( u(x, t) = \sum_{(a,b,c)\in\mathbb{Z}^3} u_{a,b,c} \exp \left( 2\pi i \left[ \frac{a}{\alpha} x_1 + \frac{b}{\beta} x_2 + \frac{c}{\gamma} t \right] \right) \)

Substituting it into (2) and comparing coefficients, we have

\[
u_{a,b,c} = \frac{\gamma^2}{4\pi^2} \frac{f_{a,b,c}}{a^2 \frac{\gamma^2}{\alpha^2} + b^2 \frac{\gamma^2}{\beta^2} - c^2}
\]

For \( u \) to be smooth it suffices to verify that

\[
\left| a^2 \frac{\gamma^2}{\alpha^2} + b^2 \frac{\gamma^2}{\beta^2} - c^2 \right| \geq C \max\{|a|, |b|\}^{-\tau},
\]

for some \( C > 0, \tau > 1 \) for all \((a, b, c)\in\mathbb{Z}^3\) with \(a \neq 0\). That means, this condition can only fail if for all \(\tau > 1\) the inequality

\[
\left| a^2 \frac{\gamma^2}{\alpha^2} + b^2 \frac{\gamma^2}{\beta^2} - c^2 \right| < \max\{|a|, |b|\}^{-\tau},
\]

holds for infinitely many \((a, b, c)\in\mathbb{Z}^3\) with \(a \neq 0\).
A general Diophantine problem

Consider a general differential operator

\[
\frac{\partial^p}{\partial t^p} - \frac{\partial^n}{\partial x_1^n} - \frac{\partial^m}{\partial x_2^m}.
\]

For any triple \((n, m, p) \in \mathbb{N}^3\) and any approximating function \(\psi\) define \(W_{n,m}(\psi)\) to be the set of vectors \(x = (x_1, x_2) \in [0, 1)^2\) for which the inequality

\[
|a^n x_1 + b^m x_2 - c^p| < \psi(h_{a,b})
\]

holds for infinitely many \((a, b, c) \in \mathbb{N}^2 \times \mathbb{Z}_{\geq 0}\).

Here, we have assigned a natural height \(h_{a,b} := \max(a^n, b^m)\) to each pair \((a, b)\) of positive integers.

How big is \(W_{n,m}(\psi)\)?

How big is \(Bad_{n,m}(\psi)\)?
How big is $Bad_{n,m}^p(\psi)$?

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A Khintchine Type Theorem

Theorem (Harrap–Hussain–Kristensen, 2015)

Assume that either \( n = m = 1 \) or \( \gcd(n, m) \geq 2 \). Then, for every approximating function \( \psi \) we have that

\[
\left| W_{n,m}^p(\psi) \right| = \begin{cases} 
0, & \sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_a^1 h_b^{1-1/p}} < \infty. \\
1, & \sum_{(a,b) \in \mathbb{N}^2} \frac{\psi(h_{a,b})}{h_a^1 h_b^{1-1/p}} = \infty.
\end{cases}
\]
A Jarnik Type Theorem

Theorem (Harrap–Hussain–Kristensen, 2015)

Let $\psi$ be an approximating function and assume that either $n = m = p = 1$ or $\gcd(n, m) \geq 2$. Let $f$ be a dimension function such that $r^{-2}f(r)$ is monotonic and for notational convenience let $g : r \to r^{-1}f(r)$ be another dimension function. Then,

$$
\mathcal{H}^f \left( W_{n,m}^p(\psi) \right) = \begin{cases} 
0, & \sum_{(a,b) \in \mathbb{N}^2} g \left( \frac{\psi(h_{a,b})}{h_{a,b}} \right) h_{a,b}^{1/p} < \infty. \\
\mathcal{H}^f \left( [0,1)^2 \right), & \sum_{(a,b) \in \mathbb{N}^2} g \left( \frac{\psi(h_{a,b})}{h_{a,b}} \right) h_{a,b}^{1/p} = \infty.
\end{cases}
$$
Corollary

Assume that either \( n = m = p = 1 \) or \( \gcd(n, m) \geq 2 \). Let \( \tau > 1 \), then
\[
\dim_H \left( \mathcal{W}_{n,m}^p \left( \psi : r \to r^{-\tau} \right) \right) = 1 + \min \left\{ 1, \frac{1}{n} + \frac{1}{m} + \frac{1}{p} \right\}.
\]

It also implies for example that \( \mathcal{H}^s \left( \mathcal{W}_{n,m}^p \left( \psi : r \to r^{-\tau} \right) \right) = \infty \) at the critical exponent \( s = \dim_H \left( \mathcal{W}_{n,m}^p \left( \psi : r \to r^{-\tau} \right) \right) \).

Corollary

If \( f \) is smooth and periodic in \( x_1, x_2, t \) with periods \( \alpha, \beta \) and \( \gamma \) respectively then the given PDE is solvable with \( u \) smooth and periodic with the same periods whenever \( (\gamma^2 / \alpha^2, \gamma^2 / \beta^2) \) does not belong to
\[
\bigcap_{\tau > 1} \mathcal{W}_{n,m}^p \left( \psi : r \to r^{-\tau} \right),
\]
a null set of Hausdorff dimension 1.
Sketch of the proof: the convergence case

(Borel–Cantelli Lemma)

\[ y = \left( -\frac{a^n}{b^m} \right)x + \frac{c^p}{b^m} \]

[Graph of line]

\( \ell_{a,b}(c) \)

\[ \text{width} \approx \frac{\psi(h_{a,b})}{\sqrt{a^{2n} + b^{2m}}} \]

(Hausdorff–Cantelli lemma)

\[ y = \left( -\frac{a^n}{b^m} \right)x + \frac{c^p}{b^m} \]

[Graph of line]

\[ \text{width} \approx f \left( \frac{\psi(h_{a,b})}{h_{a,b}} \right) \]
Lemma

Let $\Omega$ be an open subset of $\mathbb{R}^t$ and let $E$ be a Borel subset of $\mathbb{R}^t$. If there exist strictly positive constant $r_0$ such that for any ball $B$ in $\Omega$ of radius $r(B) < r_0$ we have

$$|E \cap B| \gg |B|,$$

where the implied constant is independent of $B$, then $E$ has full measure in $\Omega$.  

(3)
Lemma ((Second) Borel–Cantelli)

Let $E_t$ be a sequence of measurable sets which are **quasi-independent on average**; that is, the sequence $E_t$ satisfies

$$\sum_{t=1}^{\infty} |E_t| = \infty$$

(4)

and there exists some strictly positive constant $\alpha$ for which

$$\sum_{s,t=1}^{Q} |E_s \cap E_t| \leq \frac{1}{\alpha} \left( \sum_{t=1}^{Q} |E_t| \right)^2$$

for all $Q \geq 1$. Then,

$$\left| \limsup_{t \to \infty} E_t \right| \geq \alpha.$$
Estimating the measure of $\ell_{a,b} \cap B$

\[
\left| \ell_{a,b} \cap B \right| \asymp |B| \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}}.
\]

Thus

\[
\sum_{(a,b) \in \mathbb{N}^2} \left| \ell_{a,b} \cap B \right| \asymp \sum_{(a,b) \in \mathbb{N}^2} |B| \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty.
\]
Estimating the measure of $\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B$

$$|\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B| \leq |B| \frac{\psi(h_1)\psi(h_2)}{h_1^{1-1/p} h_2^{1-1/p}} \cdot \left(1 + \frac{1}{r h_1^{1/p} \sin \alpha}\right). \quad (6)$$

We now split our calculation into three exhaustive subcases depending on the size of the angle $\alpha$.

1. $\sin \alpha \geq \frac{1}{r h_1^{1/p}}. \quad (7)$

$\frac{1}{r h_1^{1/p}} \geq \sin \alpha \geq \frac{1}{r^{1+\frac{k}{M}} h_1^{\frac{k}{pN}} h_2^{\frac{1}{p}}}, \quad (8)$

3. $\sin \alpha < r^{-\left(1+\frac{k}{M}\right)} h_1^{-\frac{k}{pN}} h_2^{-\frac{1}{p}}. \quad (9)$
Estimating the measure of $\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B$
Number Theory Down Under

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