Theory and Applications of Optimal Control Problems with Time-Delays

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EXPECT DELAYS

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Optimal Control Problems with Time-Delays
Theory and Numerics for non-delayed optimal control problems with control and state constraints are rather complete:

1. Necessary and sufficient conditions,
2. Stability and sensitivity analysis,
3. Numerical methods:
   - Boundary value methods, Discretization and NLP,
   - Semismooth Newton methods,
4. Real-time control techniques for perturbed extremals.

**Challenge:** establish similar theoretical and numerical methods for delayed (retarded) optimal control problems.
This book is devoted to the theory and applications of second-order necessary and sufficient optimality conditions in the calculus of variations and optimal control. The authors develop theory for a control problem with ordinary differential equations subject to boundary conditions of both the equality and inequality type and for mixed state-control constraints of the equality type. The book is distinctive in that:

- necessary and sufficient conditions are given in the form of no-gap conditions,
- the theory covers broken extremals where the control has finitely many points of discontinuity, and
- a number of numerical examples in various application areas are fully solved.

This book is suitable for researchers in calculus of variations and optimal control and researchers and engineers in optimal control applications in mechanics, mechatronics, physics, chemical, electrical, and biological engineering, and economics.

Nikolai P. Osmolovskii is a Professor in the Department of Informatics and Applied Mathematics, Moscow State Civil Engineering University; the Institute of Mathematics and Physics, University of Siedlce, Poland; the Systems Research Institute, Polish Academy of Science; the University of Technology and Humanities in Radom, Poland; and of the Faculty of Mechanics and Mathematics, Moscow State University. He was an Invited Professor in the Department of Applied Mathematics, University of Bayreuth, Germany (2000), and at the Centre de Mathématiques Appliquées, École Polytechnique, France (2007). His fields of research are functional analysis, calculus of variations, and optimal control theory. He has written fifty papers and four monographs.

Helmut Maurer was a Professor of Applied Mathematics at the Universität Münster, Germany (retired 2010) and has conducted research in Austria, France, Poland, Australia, and the United States. His fields of research in optimal control are control and state constraints, numerical methods, second-order sufficient conditions, sensitivity analysis, real-time control techniques, and various applications in mechanics, mechatronics, physics, biomedical and chemical engineering, and economics.

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Overview

1. Formulation of optimal control problems with state and control delays.
2. Example: Two-stage continuous stirred tank reactor (CSTR).
4. NLP methods: discretize and optimize.
5. Optimal control of the innate immune response.
Delayed Optimal Control Problem with State Constraints

State \( x(t) \in \mathbb{R}^n \), Control \( u(t) \in \mathbb{R}^m \), Delays \( d_x, d_u \geq 0 \).

**Dynamics and Boundary Conditions**

\[
\dot{x}(t) = f(x(t), x(t - d_x), u(t), u(t - d_u)), \quad \text{a.e. } t \in [0, t_f],
\]

\[
x(t) = x_0(t), \quad t \in [-d_x, 0],
\]

\[
u(t) = u_0(t), \quad t \in [-d_u, 0),
\]

\[
\psi(x(t_f)) = 0
\]

**Control and State Constraints**

\[
u(t) \in U \subseteq \mathbb{R}^m, \quad S(x(t)) \leq 0, \quad t \in [0, t_f] \quad (S : \mathbb{R}^n \to \mathbb{R}^k).
\]

**Minimize**

\[
J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), x(t - d_x), u(t), u(t - d_u)) \, dt
\]

All functions are assumed to be sufficiently smooth.
Two-Stage Continuous Stirred Tank Reactor (CSTR)

Time delays are caused by transport between the two tanks.
A chemical reaction $A \Rightarrow B$ is processed in two tanks.

**State and control variables:**

**Tank 1:**
- $x_1(t)$: (scaled) concentration
- $x_2(t)$: (scaled) temperature
- $u_1(t)$: temperature control

**Tank 2:**
- $x_3(t)$: (scaled) concentration
- $x_4(t)$: (scaled) temperature
- $u_2(t)$: temperature control
Dynamics of the Two-Stage CSTR

Reaction term in Tank 1 : \( R_1(x_1, x_2) = (x_1 + 0.5) \exp\left(\frac{25x_2}{1+x_2}\right) \)

Reaction term in Tank 2 : \( R_2(x_3, x_4) = (x_3 + 0.25) \exp\left(\frac{25x_4}{1+x_4}\right) \)

Dynamics:

\[
\begin{align*}
\dot{x}_1(t) &= -0.5 - x_1(t) - R_1(t), \\
\dot{x}_2(t) &= -(x_2(t) + 0.25) - u_1(t)(x_2(t) + 0.25) + R_1(t), \\
\dot{x}_3(t) &= x_1(t - d) - x_3(t) - R_2(t) + 0.25, \\
\dot{x}_4(t) &= x_2(t - d) - 2x_4(t) - u_2(t)(x_4(t) + 0.25) + R_2(t) - 0.25.
\end{align*}
\]

Initial conditions:

\[
\begin{align*}
x_1(t) &= 0.15, & x_2(t) &= -0.03, & -d \leq t \leq 0, \\
x_3(0) &= 0.1, & x_4(0) &= 0.
\end{align*}
\]

Delays \( d = 0.1, \ d = 0.2, \ d = 0.4 \) in the state variables \( x_1, x_2 \).
Optimal control problem for the Two-Stage CSTR

Minimize \( \int_0^{t_f} (x_1^2 + x_2^2 + x_3^2 + x_4^2 + 0.1u_1^2 + 0.1u_2^2) \, dt \quad (t_f = 2) \).

Hamiltonian with \( y_k(t) = x_k(t - d) \), \( k = 1, 2 \):

\[
H(x, y, \lambda, u) = f_0(x, u) + \lambda_1 \dot{x}_1 \\
+ \lambda_2(-(x_2 + 0.25) - u_1(x_2 + 0.25) + R_1(x_1, x_2)) \\
+ \lambda_3(y_1 - x_3 - R_2(x_3, x_4) + 0.25) \\
+ \lambda_4(y_2 - 2x_4 - u_2(x_4 + 0.25) + R_2(x_3, x_4) + 0.25)
\]

Advanced adjoint equations:

\[
\dot{\lambda}_1(t) = -H_{x_1}(t) - \chi_{[0,t_f-d]} \lambda_3(t + d), \\
\dot{\lambda}_2(t) = -H_{x_2}(t) - \chi_{[0,t_f-d]} \lambda_4(t + d), \\
\dot{\lambda}_k(t) = -H_{x_k}(t) \quad (k = 3, 4).
\]

The minimum condition yields \( H_u = 0 \) and thus

\[
u_1 = 5\lambda_2(x_2 + 0.25), \quad u_2 = 5\lambda_4(x_4 + 0.25).
\]
Two-Stage CSTR with free $x(t_f)$: $x_1$, $x_2$, $x_3$, $x_4$

Concentration $x_1$

Temperature $x_2$

Concentration $x_3$

Temperature $x_4$

Delays $d = 0.1$, $d = 0.2$, $d = 0.4$.
Two-Stage CSTR with free $x(t_f)$: $u_1, u_2, \lambda_1, \lambda_2$

Delays $d = 0.1, d = 0.2, d = 0.4$. 

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Optimal Control Problems with Time-Delays
Two-Stage CSTR with \( x(t_f) = 0 \): \( x_1, x_2, x_3, x_4 \)

Delays \( d = 0.1, \; d = 0.2, \; d = 0.4 \).
Two-Stage CSTR with $x(t_f) = 0$: $u_1, u_2, \lambda_1, \lambda_2$

**Delays** $d = 0.1, d = 0.2, d = 0.4$. 
Two-Stage CSTR with $x(t_f) = 0$ and $x_4(t) \leq 0.01$

Delays $d = 0.1$, $d = 0.2$, $d = 0.4$. 
Delayed Optimal Control Problem with State Constraints

State \( x(t) \in \mathbb{R}^n \), Control \( u(t) \in \mathbb{R}^m \), Delays \( d_x, d_u \geq 0 \).

Dynamics and Boundary Conditions

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\dot{x}(t) = f(x(t), x(t - d_x), u(t), u(t - d_u)), \quad \text{a.e. } t \in [0, t_f],
\]

\( x(t) = x_0(t), \quad t \in [-d_x, 0], \)

\( u(t) = u_0(t), \quad t \in [-d_u, 0], \)

\( \psi(x(t_f)) = 0 \)

Control and State Constraints

\( u(t) \in U \subset \mathbb{R}^m \), \( S(x(t)) \leq 0 \), \( t \in [0, t_f] \) \( (S : \mathbb{R}^n \to \mathbb{R}^k) \).

Minimize

\[
J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), x(t - d_x), u(t), u(t - d_u)) \, dt
\]
Literature on optimal control with time-delays

State delays and pure control constraints:


State and control delays and mixed control–state constraints:
Göllmann, Kern, Maurer (OCAM 2009),
Optimal control problems with state constraints

Use the transformation method of Guinn (1976) and transform an optimal control problem with delays and state constraints to a standard non-delayed optimal control problem with state constraints. Then apply the necessary conditions for non-delayed problems:

- Vinter (2000): (Nonsmooth) Optimal Control
Hamiltonian (Pontryagin) Function

\[ H(x, y, \lambda, u, v) := \lambda_0 f_0(t, x, y, u, v) + \lambda f(t, x, y, u, v) \]

- \( y \) variable with \( y(t) = x(t - d_x) \)
- \( v \) variable with \( v(t) = u(t - d_u) \)
- \( \lambda \in \mathbb{R}^n, \lambda_0 \in \mathbb{R} \) adjoint (costate) variable

Let \((u, x) \in L^\infty([0, t_f], \mathbb{R}^m) \times W^{1,\infty}([0, t_f], \mathbb{R}^n)\) be a locally optimal pair of functions. Then there exist
  - an adjoint function \( \lambda \in BV([0, t_f], \mathbb{R}^n) \) and \( \lambda_0 \geq 0 \),
  - a multiplier \( \rho \in \mathbb{R}^q \) (associated with terminal conditions),
  - a multiplier function (measure) \( \mu \in BV([0, t_f], \mathbb{R}^k) \),

such that the following conditions are satisfied for a.e. \( t \in [0, t_f] \):
Minimum Principle

(i) Advanced adjoint equation and transversality condition:

\[ \lambda(t) = \int_t^{t_f} \left( H_x(s) + \chi_{[0,t_f-d_x]}(t) H_y(s + d_x) \right) ds + \int_t^{t_f} S_x(x(s)) d\mu(s) \]

\[ + \left( \lambda_0 g + \rho \psi \right)_x(x(t_f)) \quad (\text{if } S(x(t_f)) < 0), \]

where \( H_x(t) \) and \( H_y(t + d_x) \) denote evaluations along the optimal trajectory and \( \chi_{[0,t_f-d_x]} \) is the characteristic function.

(ii) Minimum Condition:

\[ H(t) + \chi_{[0,t_f-d_u]}(t) H(t + d_u) = \min_{w \in U} \left[ H(x(t), y(t), \lambda(t), w, \nu(t)) \right. \]

\[ \left. + \chi_{[0,t_f-d_u]}(t) H(t + d_u) H(x(t + d_u), y(t), \lambda(t + d_u), u(t + d_u), w) \right] \]

(iii) Multiplier condition and complementarity condition:

\[ d\mu(t) \geq 0, \quad \int_0^{t_f} S(x(t)) d\mu(t) = 0 \]
Minimum Principle

(i) Advanced adjoint equation and transversality condition:

\[ \lambda(t) = \int_t^{t_f} (H_x(s) + \chi[0, t_f-d_x](t) H_y(s + d_x)) \, ds + \int_t^{t_f} S_x(x(s)) \, d\mu(s) \]
\[ + (\lambda_0 g + \rho \psi) x(x(t_f)) \quad (\text{if } S(x(t_f)) < 0), \]

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\[ = \min_{w \in U} \left[ H(x(t), y(t), \lambda(t), w, \nu(t)) \right. \]
\[ + \chi[0, t_f-d_u](t) H(t + d_u)H(x(t + d_u), y(t), \lambda(t + d_u), u(t + d_u), w) \]

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Minimum Principle

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+ (\lambda_0 g + \rho \psi)x(x(t_f)) \quad \text{(if} \ S(x(t_f)) < 0 \text{)},
\]

where \( H_x(t) \) and \( H_y(t + d_x) \) denote evaluations along the optimal trajectory and \( \chi_{[0,t_f-d_x]} \) is the characteristic function.

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\[
H(t) + \chi_{[0,t_f-d_u]}(t) H(t + d_u) \\
= \min_{w \in U} \left[ H(x(t), y(t), \lambda(t), w, v(t)) \\
+ \chi_{[0,t_f-d_u]}(t) H(t + d_u)H(x(t + d_u), y(t), \lambda(t + d_u), u(t + d_u), w) \right]
\]

(iii) Multiplier condition and complementarity condition:

\[
d\mu(t) \geq 0, \quad \int_0^{t_f} S(x(t)) d\mu(t) = 0
\]
Regularity conditions for \( d\mu(t) = \eta(t) dt \) if \( du = 0 \)

**Boundary arc**: \( S(x(t)) = 0 \) for \( t_1 \leq t \leq t_2 \).

**Assumption**: \( u(t) \in \text{int}(U) \) for \( t_1 < t < t_2 \).

Under certain regularity conditions we have \( d\mu(t) = \eta(t) dt \) with a smooth multiplier \( \eta(t) \) for all \( t_1 < t < t_2 \).

**Adjoint equation and jump conditions**

\[
\dot{\lambda}(t) = -H_x(t) - \chi_{[0,t_f-d_x]} H_y(t + d_x) - \eta(t) S_x(x(t)) \\
\lambda(t_k+) = \lambda(t_k-) - \nu_k S_x(x(t_k)), \quad \nu_k \geq 0
\]

at each contact or junction time \( t_k \), \( \nu_k = \mu(t_k+) - \mu(t_k-) \)

**Minimum condition on the boundary**

\[ H_U(t) = 0. \]

This condition allows to compute the multiplier \( \eta = \eta(x, \lambda) \).
Dynamic model of the immune response:


Optimal control:


Innate Immune Response: state and control variables

**State variables:**

\( x_1(t) \) : concentration of **pathogen**  
(=concentration of associated **antigen**)  
\( x_2(t) \) : concentration of **plasma cells**,  
which are carriers and producers of antibodies  
\( x_3(t) \) : concentration of **antibodies**, which kill the pathogen  
(=concentration of **immunoglobulins**)  
\( x_4(t) \) : relative characteristic of a **damaged organ**  
( 0 = healthy, 1 = dead )

**Control variables:**

\( u_1(t) \) : pathogen killer  
\( u_2(t) \) : plasma cell enhancer  
\( u_3(t) \) : antibody enhancer  
\( u_4(t) \) : organ healing factor
Innate Immune Response: state and control variables

State variables:

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Control variables:

\( u_1(t) \) : pathogen killer

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\( u_4(t) \) : organ healing factor
Generic dynamical model of the immune response

\[
\begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3A(x_4(t))x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t).
\end{align*}
\]

Immune deficiency function triggered by target organ damage

\[A(x_4) = \begin{cases} 
\cos(\pi x_4), & 0 \leq x_4 \leq 0.5 \\
0, & 0.5 \leq x_4 
\end{cases}.\]

For \(0.5 \leq x_4(t)\) the production of plasma cells stops.

State delay \(d \geq 0\) in variables \(x_1\) and \(x_3\)

Initial conditions \((d = 0)\): \(x_2(0) = 2, \ x_3(0) = 4/3, \ x_4(0) = 0\)

Case 1: \(x_1(0) = 1.5\), decay, requires no therapy (control)

Case 2: \(x_1(0) = 2.0\), slower decay, requires no therapy

Case 3: \(x_1(0) = 3.0\), diverges without control (lethal case)
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State delay \(d \geq 0\) in variables \(x_1\) and \(x_3\)

Initial conditions \((d = 0)\) : \(x_2(0) = 2,\ x_3(0) = 4/3,\ x_4(0) = 0\)

Case 1 : \(x_1(0) = 1.5\), decay, requires no therapy (control)
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Case 3 : \(x_1(0) = 3.0\), diverges without control (lethal case)
Optimal control model: cost functional

State \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \), Control \( u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \)

\( L^2 \)-functional quadratic in control: Stengel et al.

Minimize \( J_2(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)

\[ + \int_0^{t_f} (x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2) \, dt \]

\( L^1 \)-functional linear in control

Minimize \( J_1(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)

\[ + \int_0^{t_f} (x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4) \, dt \]

Control constraints: \( 0 \leq u_i(t) \leq u_{\text{max}}, \ i = 1, \ldots, 4 \)

Final time: \( t_f = 10 \)
Optimal control model: cost functional

**State** $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, \ **Control** $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$

$L^2$-functional quadratic in control: Stengel et al.

Minimize $J_2(x, u) = x_1(t_f)^2 + x_4(t_f)^2$

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Final time: $t_f = 10$
Optimal control model: cost functional

**State** \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \),  **Control** \( u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \)

**\( L^2 \)-functional quadratic in control: Stengel et al.**

Minimize \( J_2(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)

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+ \int_0^{t_f} \left( x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2 \right) dt
\]

**\( L^1 \)-functional linear in control**

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Control constraints: \( 0 \leq u_i(t) \leq u_{\text{max}}, \quad i = 1, \ldots, 4 \)

Final time: \( t_f = 10 \)
$L^2$–functional, $d = 0$ : optimal state and control variables

State variables $x_1, x_2, x_3, x_4$ and optimal controls $u_1, u_2, u_3, u_4$ : second-order sufficient conditions via matrix Riccati equation
$L^2$–functional, $d = 0$: state constraint $x_4(t) \leq 0.2$

State and control variables for state constraint $x_4(t) \leq 0.2$.

Boundary arc $x_4(t) \equiv 0.2$ for $t_1 = 0.398 \leq t \leq t_2 = 1.35$.
$L^2$–functional, multiplier $\eta(t)$ for constraint $x_4(t) \leq 0.2$

Compute multiplier $\eta$ as function of $(x, \lambda)$:

$$\eta(x, \lambda) = \lambda_2 3\pi \sin(\pi x_4) x_1 x_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4$$

Scaled multiplier 0.1$\eta(t)$ and boundary arc $x_4(t) = 0.2$
$L^2$–functional, delay $d > 0$, constraint $x_4(t) \leq \alpha$

**Dynamics with state delay $d > 0$**

\[
\begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3 \cos(\pi x_4) x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t) \\
x_4(t) &\leq \alpha \leq 0.5
\end{align*}
\]

**Initial conditions**

\[
\begin{align*}
x_1(t) &= 0, \quad -d \leq t < 0, \quad x_1(0) = 3, \\
x_3(t) &= 4/3, \quad -d \leq t \leq 0, \\
x_2(0) &= 2, \\
x_4(0) &= 0.
\end{align*}
\]
$L^2$–functional: delay $d = 1$ and $x_4(t) \leq 0.2$

State variables for $d = 0$ and $d = 1$
$L^2$–functional: delay $d = 1$ and $x_4(t) \leq 0.2$

Optimal controls for $d = 0$ and $d = 1$
Compute multiplier $\eta$ as function of $(x, \lambda)$:

$$\eta(x, y, \lambda) = \lambda_2 3\pi \sin(\pi x_4) y_1 y_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4$$

Scaled multiplier $0.1 \eta(t)$ and boundary arc $x_4(t) = 0.2$;
$\eta(t)$ is discontinuous at $t = d = 1$
Minimize

\[ J_1(x, u) = p_{11}x_1(t_f)^2 + p_{44}x_4(t_f)^2 \]
\[ + \int_0^{t_f} (p_{11}x_1^2 + p_{44}x_4^2 + q_1 u_1 + q_2 u_2 + q_3 u_3 + q_4 u_4) \, dt \]

Dynamics with delay \( d \) and control constraints

\[
\begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3A(x_4(t))x_1(t - r)x_3(t - r) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t),
\end{align*}
\]

\[ 0 \leq u_i(t) \leq u_{\text{max}}, \quad 0 \leq t \leq t_f \quad (i = 1, \ldots, 4) \]
$L^1$–functional: non-delayed problem $d = 0: u_{\text{max}} = 2$
$L^1$–functional : delayed problem $d = 1 : u_{\text{max}} = 2$
$L^1$–functional: non-delayed, time–optimal control for

\[ x_1(t_f) = x_4(t_f) = 0, \quad x_3(t_f) = \frac{4}{3} \]

\[ u_{\text{max}} = 1: \text{minimal time} \quad t_f = 2.2151, \text{ singular arc for} \quad u_4(t) \]
Further applications and future work

1. Optimal oil extraction and exploration: state delay (Bruns, Maurer, Semmler)
2. Biomedical applications: optimal protocols in cancer treatment and immunology
3. Vintage control problems
4. Delayed control problems with free final time
5. Optimal control problems with state-dependent delays
6. Verifiable sufficient conditions
Thank you for your attention!