Origami Mathematics in Education

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Tools and Mathematics
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Origami

• The Art of Folding

http://www.jccc.on.ca/assets/images/origami5.jpg
Origami

- The Art of Folding

http://img.gawkerassets.com/img/17jp3vs9qkjb6jpg/original.jpg
http://res.artnet.com/news-upload/2014/05/origami-6.jpg
http://www.joostlangeveldorigami.nl/fotos/historyoforigami/bug.jpg
http://i.ytimg.com/vi/5nZtibCqFxw/hqdefault.jpg
Origami

- The Art of Folding
Origami

- The Art of Folding
Origami in the Classroom
1D Origami
Folding In Half

- How many times can you fold paper in half?
  - 8 times?
Folding In Half

• How many times can you fold paper in half?
  – 8 times?

• Is there an upper limit?
Folding In Half

• Britney Gallivan 2001

\[ L = \frac{\pi \cdot t}{6} \cdot (2^n + 4)(2^n - 1) \]

\[ W = \pi \cdot t \cdot 2^{3(n-1)/2} \]
Activity 1
Parabolas

• Why does it work?
• Can other conics be constructed?
• What if you use non-flat paper?
• What can we learn concerning:
  – Parabolas?
  – Envelopes?
  – Derivatives?
  – Tangents?
  – Convergence of sequences?
Knots
Knots

(a)

(b)

(c) $a=b$

(d)
Knots
Knots

(a)

(b)

(c)

(d)
Knots

(a) 

(b) 

(c)
Knots

\[ 2 \cos(2\pi - \theta) = \frac{1}{2} \]
Knots

Crease: 1st 2nd 3rd

M: Mountain
V: Valley
U: Upper
M: Middle
L: Lower

Mirror Image

Lower ← Upper
Knots
Knots

- Explorations:
  - Perimeter, area
  - Irregular patterns
  - Enumerations
  - Knot theory, topology
Activity 2

- Fujimoto approximation
Fujimoto Approximation

- Error is halved at each operation
- Repeating left-right pattern represented as the binary expansion of $1/n$
  - $1/5: \ 0.00110011\ldots$
  - $1/7: \ 0.011011011\ldots$
Between 1D and 2D
Origami Constructions

- What geometric constructions are possible?
Origami Constructions

(O1) Given two points $p_1$ and $p_2$, we can fold a line connecting them.

(O2) Given two points $p_1$ and $p_2$, we can fold $p_1$ onto $p_2$.

(O3) Given two lines $l_1$ and $l_2$, we can fold line $l_1$ onto $l_2$.

(O4) Given a point $p_1$ and a line $l_1$, we can make a fold perpendicular to $l_1$ passing through the point $p_1$.

(O5) Given two points $p_1$ and $p_2$ and a line $l_1$, we can make a fold that places $p_1$ onto $l_1$ and passes through the point $p_2$.

(O6) Given two points $p_1$ and $p_2$ and two lines $l_1$ and $l_2$, we can make a fold that places $p_1$ onto line $l_1$ and places $p_2$ onto line $l_2$.

(O7) Given a point $p_1$ and two lines $l_1$ and $l_2$, we can make a fold perpendicular to $l_2$ that places $p_1$ onto line $l_1$. 
Origami Constructions

(A1) $F_{LF}(P_1) \leftrightarrow P_2$

(A2) $F_{LF}(L_1) \leftrightarrow L_2$

(A3) $F_{LF}(L) \leftrightarrow L$

(A4) $F_{LF}(P) \leftrightarrow L$

(A5) $L_F \leftrightarrow P$
Origami Constructions

• 22.5 degree angle restriction
  - All coordinates of the form $\frac{m+n\sqrt{2}}{2^l}$ are constructible
  - Algorithm linear in $l$, $\log(m)$, $\log(n)$
Origami Constructions

• More generally:
  - Constructible numbers of the form $2^m 3^n$
  - Angle trisection, cube doubling possible
  - Roots of the general cubic
Origami Constructions

- Polynomial root finding, Lill's method

\[ x^4 - a_3 x^3 + a_2 x^2 - a_1 x - a_0 = 0 \]

\[ x^2 - a_1 x - a_0 = 0 \]

\[ x^3 - a_2 x^2 + a_1 x - a_0 = 0 \]
Origami Constructions

(AL1) $F_{LF_a}(L_{F_b}) \leftrightarrow L_{F_b}$

(AL2) $F_{LF_a}(L) \leftrightarrow L$

(AL3) $L_{F_a} \leftrightarrow P$

(AL4) $F_{LF_a}(L) \leftrightarrow L_{F_b}$

(AL5) $F_{LF_a}(P) \leftrightarrow L_{F_b}$

(AL6) $F_{LF_a}(P) \leftrightarrow L$

(AL7) $F_{LF_a}(P) \leftrightarrow F_{LF_b}(L)$

(AL8) $F_{LF_a}(P_1) \leftrightarrow F_{LF_b}(P_2)$

(AL9) $F_{LF_a}(L_1) \leftrightarrow F_{LF_b}(L_2)$

(AL10) $F_{LF_b}(P_{LF_a,L_1}) \leftrightarrow L_2$
Origami Constructions

- 489 distinct two-fold line constructions
Origami Constructions

- General quintic construction
Origami Constructions

- Higher order equations, real solutions
  - Order $n$ requires $(n-2)$ simultaneous folds
- What can we learn concerning:
  - Polynomial roots
  - Geometric constructions
  - Field theory
  - Galois theory
2D Folding
Flat Foldability Theorems

- Maekawa's theorem: $|M-V|=2$, even degrees

[Image: http://en.wikipedia.org/wiki/Maekawa%27s_theorem#/media/File:Kawasaki%27s_theorem.jpg]
Flat Foldability Theorems

- **Kawasaki's theorem**: sum of alternating angles equals 180°
Flat Foldability Theorems

- Crease patterns are two-colorable

Flat Foldability is Hard

- Deciding flat-foldability is NP-complete

Circle Packing

- What is a flap?
Circle Packing

- What is a flap?

Origami Design Secrets, by Robert Lang
Circle Packing

- Understanding crease patterns using circles
Circle Packing

- Design algorithm
  - Uniaxial tree theory
  - Universal molecule

*Origami Design Secrets*, by Robert Lang
Circle Packing

$r = 0.354$

$r = 0.324$
Circle Packing

- $N = 1$
  - $r = 1.000$

- $N = 2$
  - $r = 0.707$

- $N = 3$
  - $r = 0.518$

- $N = 4$
  - $r = 0.500$

- $N = 5$
  - $r = 0.354$

- $N = 6$
  - $r = 0.300$

- $N = 7$
  - $r = 0.270$

- $N = 8$
  - $r = 0.259$

- $N = 9$
  - $r = 0.250$
Circle Packing

$N=1$

$N=2$

$N=3$

$N=4$

$N=5$

$N=6$

$N=7$

$N=8$

$N=9$
Circle Packing
Circle Packing
Circle Packing

- Software TreeMaker automates solving the circle packing problem
- Non-linear constrained optimization problem
Coloring Problems

- Miura-ori: row staggered pattern
- One angle parameter
Coloring Problems

- Miura-ori: 3-colorings of the square lattice
- Equivalent to an ice problem in statistical mechanics
- Asymptotic number of colorings is \((4/3)^{3N/2}\)
Beyond Flat 2D origami
Fractal Origami
Fractal Origami
Fractal Origami
Breaking Flat Foldability
Breaking Flat Foldability
Breaking Flat Foldability
Breaking Flat Foldability
Breaking Flat Foldability

• Explorations:
  – Surface Area
  – Volume
  – Optimization problem
  – Other shapes
Non-flat paper
Non-flat paper

- Conics
Non-flat paper

- Spherical paper, hyperbolic paper
  - One fold constructions are known
Curved Folding
Curved Folding
Curved Folding
Curved Folding
Curved Folding
Curved Folding

- No systematic algorithm for design known
- Direct applications in differential geometry
- Curved folding on non-flat paper not yet explored
A World Of Origami Maths

- Areas of mathematics involved only limited by imagination
- Many more applications in textbooks and convention proceedings
- Many simple research projects are awaiting students and teachers
Thank You!