Phase Plotting for Hyperbolic Geometry

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Phase plotting is a way of visualizing complex functions \( f : \mathbb{C} \rightarrow \mathbb{C} \).

Where \( f(r_1 e^{i\theta_1}) = r_2 e^{i\theta_2} \), we plot the domain space, coloring points according to argument of image \( \theta_2 \).

- Top right: \( z \rightarrow z \).
- Bottom right: \( z \rightarrow z^3 \).
Some Examples

Figure: Left to right: $\sinh(z), z \cdot e^z, \zeta(z)$. 
Recapturing the Modulus

- We can also plot in 3d to recapture the modulus information.
- Let \( f(r_1 e^{i\theta_1}) = r_2 e^{i\theta_2} \)
- Again we plot over the domain space, coloring points according to argument of image \( \theta_2 \)
- We also give them vertical height corresponding to their modulus \( r_2 \).

**Figure:** Phase plots with modulus included. Left: \( z \rightarrow z \), Right: \( z \rightarrow z^2 \).
Phase plotting is a relatively new tool.

Recent attention
- “Complex Beauties” annual calendar (of which Jonathan Borwein was quite fond) [3]

Wegert’s Matlab code is available for download on his site.

Figure: Left: Elias Wegert’s “Visual Complex Functions.” Right: “Moment function of a 4-step Pplanar random walk” by Jonathan M. Borwein and Armin Straub from 2016 Complex Beauties calendar.
Differential Geometry

- Conformal Mappings are mappings which preserve the angles at which lines meet (and signs thereof).
- Direct Motions are mappings such that the distance between points is equal to the distance between their images.
- Parallel axiom: for a line $L$ and point $p$ there exists exactly one line through $p$ which doesn’t intersect $L$.
- Geometries which do not obey the parallel axiom:
  - Spherical Geometry (no lines through $p$)
  - Hyperbolic Geometry (more than one line through $p$)
  - Both have constant curvature (intrinsc property)
- The type of geometry determines how many types of direct motions there are.
- This is because conformal maps can be expressed as compositions of reflections across lines.
Phase plotting on the Riemann sphere has already been employed by Wegert.

Figure: Left: The construction of the Riemann Sphere with stereographic projection. Right: phase plotting for a Möbius transformation (direct motion) on the Riemann Sphere.
We extend the notion of phase plotting to surfaces useful in visualizing hyperbolic geometry:

- Pseudosphere
- Poincaré Disc
- Beltrami Half-Sphere
- Klein Disc

For the task, we had to redefine the hsv coloring rules for different representations of hyperbolic space.

We did so using Maple.

- We exploited Maple’s texture plotter in order to cover 3d objects with colors.
- This generates much nicer shapes than simply coloring individual points in space.
**Figure:** The conformal map from the pseudosphere to the hyperbolic upper half plane.
The map is between the pseudosphere and a small area of the upper half plane.

If we colored according to the planar phase plotting rules, problems:

- Fewer colors for visualization
- Coloring would be tied to Euclidean geometry rather than Hyperbolic geometry, warping perspective.
- Unable to tell if points mapped out of visible region.

Solution: defined a new coloring scheme unique to hyperbolic space.

**Figure:** Colors change along tractrices rather than Euclidean subspaces.
Figure: Computing a direct motion (h-rotation) in hyperbolic space. Here $M = I_{L_2} \circ I_{L_1}$ where $L_1$ and $L_2$ correspond to circles in $\mathbb{C}$ centered at 0 and $2\pi$ with radius $2\pi$. The Möbius transformation is $4 \times \pi^2/(2 \times \pi - z)$. 
Pseudosphere: h-rotation

- The regions sent out of view are the regions we expected to be sent out of view.
- The rainbow spectrum is now rotated, as hyperbolic space has been rotated.
- Notice how non-tractrix lines are now visible!
Figure: Direct motion: a limit rotation. \( M = I_{L_2} \circ I_{L_1} \) where \( L_1 \) and \( L_2 \) correspond to circles in \( \mathbb{C} \) centered at \( \frac{3}{2} \pi \) and \( 2\pi \) with radii \( \frac{1}{2} \pi, \pi \) respectively. The Möbius transformation is \( \pi^2/(-z + 2\pi) \).
Pseudosphere: limit rotation

- The center of the rotation is at the right rear.
- Much of foreground is green; these points have all been pulled towards the right rear.
- Only some points starting inside the circle for $L_1$ are mapped to the left rear.
Figure: Direct motion: an h-translation. $M = I_{L_2} \circ I_{L_1}$ where $L_1$ and $L_2$ correspond to circles in $\mathbb{C}$ centered at $\pi$ with radii $\frac{1}{3}\pi$, $\frac{1}{2}\pi$ respectively. The Möbius transformation is $\frac{9}{4}z - \frac{5}{4}\pi$. 
Pseudosphere: limit rotation

- We see tractrices sent to tractrices.
- Some points are translated out of view.
- Space appears to contract, but has not actually done so. If we made our translation by reflecting across tractrix lines, this effect would not be visible.
One can use a simple “hack” of the interface to determine the tractrix height of the image points.

Simply compose the map:

$$F(z) = \frac{2\pi}{\alpha} \cdot \log(\Re(z)) + \exp(1) \cdot i$$

on the motion in question.

Here the color spectrum begins at tractrix edge; $\alpha$ is chosen to be the tractrix height at which it terminates.

**Figure:** $F$ composed on identity map where $\alpha = 2$. 
Figure: Left: Construction of the Poincaré Disc. Right: phase plotting on Poincaré Disc as defined by our rule.
We adopt a new plotting rule.

Still colors tractrix generators in a single color:
- Pre-images of h-lines are still h-lines
- Consistent with Pseudosphere

The trick is subtle.

Where $T$ is inversion map $f$ user function, HSV map for $p$ in disc is:

$$\frac{1}{2\pi} \arg \circ T \circ \Re \circ f \circ T.$$
Figure: Left: $h$-translation $z \to z + 2$. Right: $h$-rotation $z \to (4\pi^2)/(2\pi - z)$ corresponding to inversion in 2 circles radius $2\pi$ centered at $0, 2\pi$. Notice that the preimages of lines are still lines.
**Figure:** Left: a *limit* rotation \( z \rightarrow \pi (2\pi - 3z)/ (\pi - 2z) \) corresponds to inversion in circles of radius \( \pi \) centered at 0 and \( 2\pi \). Right: a map which is not a direct motion: \( z \rightarrow z^3 \).
Figure: Two more maps which are not direct motions. Left: $z \to sinh(z)$. Right: $z \to sin(z)$. 
Beltrami Half-Sphere

Figure: Left: Construction of the Beltrami Half-Sphere (first step) and Klein Disc (second step). Right: phase plotting on Beltrami Half-Sphere for $z \rightarrow z$.

- The Beltrami half-sphere is constructed via a lower stereographic projection of the Poincare disc.
- Phase plotting rule is inherited from Poincare disc.
Figure: Left: h-translation $z \rightarrow z - 2$. Right: h-rotation $z \rightarrow (z - 3)/(z - 1)$ corresponding to inversion in circles of radius 2 centered at $-1, 1$.

- Notice how lines in hyperbolic space are now semi-circles orthogonal to unit circle.
- Hyperbolic subspaces are hemispheres orthogonal to unit circle.
Figure: Left: a limit rotation \( z \rightarrow z/(z + 1) \) corresponding to inversion in two circles of radius 2 centered at \(-2, 2\). Right: a map which is not a direct motion: \( z \rightarrow z^3 \).
Klein Disc

Figure: Left: Construction of the Beltrami Half-Sphere (first step) and Klein Disc (second step). Right: phase plotting on Klein Disc for $z \rightarrow z$.

- The Klein Disc.
- Phase plotting rule is again inherited from Poincaré disc.
Figure: Left: h-translation $z \rightarrow z + 2$. Right: h-rotation $z \rightarrow (z - 3)/(z - 1)$ corresponding to inversion in circles of radius 2 centered at $-1, 1$. 

Figure: Left: a \textit{limit} rotation $z \rightarrow z/(z + 1)$ corresponding to inversion in two circles of radius 2 centered at $-2, 2$. Right: a map which is not a direct motion: $z \rightarrow z^3$. 
Figure: Two more maps which are not direct motions. Left: $z \rightarrow sinh(z)$. Right: $z \rightarrow sin(z)$.
