1. The travelling wave solution of the KdV equation written as a first order system is

\[
\frac{dv}{d\zeta} = cf - 3f^2 \\
\frac{df}{d\zeta} = v
\]

(a) Show that the equilibrium point at (0, 0) is a saddle and sketch the solution near it.
(b) Draw the phase portrait using \textit{pplane} or equivalent software for the system and show on this the homoclinic connection and a periodic solution.
(c) Solve the equation numerically and plot the solitary wave solution and cnoidal wave solution as a function of $\zeta$ corresponding the curves above. Hand in both the MATLAB code and the plots.

[10 marks]

2. Solve the following linear pde’s equations numerically using a spectral method. Draw the solution as a function of time. In all case hand in both the MATLAB code and the plots.

(a) \[
\partial_t u + \partial_x^3 u / 10 = 0 \\
u(x, 0) = \exp(-x^2)
\]

(b) \[
\partial_t u + \partial_x u = 0 \\
u(x, 0) = \exp(-x^2)
\]

(c) \[
\partial_t u + \partial_x u + \partial_x^3 u / 10 = 0 \\
u(x, 0) = \exp(-x^2)
\]

[5 marks]

3. Solve the KdV using a slit step spectral method for the following initial conditions and draw the solutions as a function of time. In all case hand in both the MATLAB code and the plots.

(a) \[
u(x, 0) = \exp(-x^2)
\]

(b) \[
u(x, 0) = 8 \text{sech}^2(2(x + 8)) + 2\text{sech}^2((x + 5))
\]
4. We can stabilise the traffic wave equations by adding a small amount of diffusion. The modified equation is
\[
\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2}
\]
We will choose a small value for \( \nu = 0.001 \). Use the standard relationship \( v = 1 - \rho \) to eliminate \( q \) and then solve the equation numerically using a split step spectral method, for the initial condition of assignment 1 question 6
\[
\rho(x,0) = \frac{1}{2} + xe^{-x^2}
\]
Draw the solutions as a function of time. In hand in both the MATLAB code and the plots. How do the solutions compare to those found in the previous assignment.

5. For the following equations, derive a conservation law.
(a) \( \partial_t u + \partial_x^2 u + \partial_x u \cos u = 0 \)
(b) \( u \partial_t u + (\partial_x u) u^2 = 0 \)
(c) The modified KdV
\[ \partial_t v - 3 \partial_x (v^3) + \partial_x^3 v = 0 \]

6. In class we derived the first four conservation laws for the KdV. Derive the fifth

7. For the delta function potential \( u(x,0) = u_0 \delta(x) \) we found that there is a single negative (bound) eigenfunction denoted by
\[
w_1(x) = \begin{cases} \sqrt{k_1} e^{k_1 x}, & x < 0 \\ \sqrt{k_1} e^{-k_1 x}, & x > 0 \end{cases}
\]
where \( k_1 = u_0/2 \). Find the solution to the integral equation
\[ K(x,y;t) + F(x+y;t) + \int_x^\infty K(x,z;t) F(z+y;t) \, dz = 0 \]
if we approximate the function \( F \) by
\[ F(x,t) = k_1 e^{8k_1^2 t} e^{-k_1 x} \].
Assume that we can write \( K \) in the form
\[ K(x,y,t) = -v_1(x,t) e^{-k_1 y} \].
Find the corresponding function \( u(x,t) \) and write \( u \) in the standard soliton form.
8. Find the eigenvalues of the bound states (find the negative eigenvalues) for the potential

\[ u(x, 0) = u_0 \delta (x - 1) + u_0 \delta (x + 1). \]

Show that when \( u_0 \) is large there are two solutions with \( k = -\sqrt{\lambda} \approx u_0/2 \) and that for \( u_0 \) small there is one solution with \( k = -\sqrt{\lambda} \approx u_0 \).

[10 marks]
9. Consider the potential given by

\[ u = \begin{cases} \frac{u_0}{\varepsilon}, & x \in [-\varepsilon, \varepsilon] \\ 0, & x \notin [-\varepsilon, \varepsilon] \end{cases} \]

(a) Show that for \( \varepsilon \) small enough there is a single negative bound eigenfunction.

(b) Show that as \( \varepsilon \) tends to zero the eigenvalue \(-\lambda = k^2\) tends to \( k = u_0/2 \) (the eigenfunction for the corresponding delta function potential).

(c) For small \( \varepsilon \) show that we can write \(-\lambda = k^2\) where

\[ k = u_0/2 + \varepsilon \Delta k \]

and determine \( \Delta k \).

[10 marks]

10. Consider the KdV with initial condition

\[ u(x, 0) = \begin{cases} 0, & x \notin [-1, 1], \\ 30, & -1 < x < 1. \end{cases} \]

(a) Find the eigenvalues of the bound states. (find the negative eigenvalues).

(b) Calculate the solution numerically and show that the bound states correspond to the soliton solutions. Remember that you will need to smooth the solution.

[10 marks]

Total Marks: 75