Burgers Equation

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Introduction

We have already met the conservation law for the traffic equations

$$\partial_t \rho + c(\rho) \partial_x \rho = 0$$

and seen how this leads to shocks. We can smooth this equation by adding dispersion to the equation to give us

$$\partial_t \rho + c(\rho) \partial_x \rho = \nu \partial_x^2 \rho$$

where $\nu > 0$

The simplest equation of this type is to write

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u$$

(changing variables to $\eta$ and this equation is known as Burgers equation.

Travelling Wave Solution

We can find a travelling wave solution by assuming that

$$u(x, t) = u(x - ct) = u(\xi)$$

This leads to the equations

$$-cu' + u' u - \nu u'' = \zeta$$

We begin by looking at the phase plane for this system, writing $\omega = u'$ so that

$$\frac{du}{d\xi} = w$$

$$\frac{d\omega}{d\xi} = \frac{1}{\nu} (w(u - c))$$

This is a degenerate system with the entire $\eta$ axis being equilibria.

We can also solve this equation exactly as follows.

$$-cu' + u' u - \nu u'' = \zeta$$

can be integrated to give

$$-cu + \frac{1}{2} u^2 - \nu u' = c_1$$

which can be rearranged to give

$$u' = \frac{1}{2\nu} \left( (u)^2 - 2cu - 2c_1 \right)$$

We define the two roots of the quadratic $(u)^2 - 2nu - 2c_1 = 0$ by $u_1$ and $u_2$ and we assume that $u_2 < u_1$. Note that there is only a bounded solution if we have two real roots and for the bounded solution $u_2 < u < u_1$. We note that the wave speed is
The equation can therefore be written as
\[ 2\nu u' = (u - u_1)(u - u_2) \]
which has solution
\[ u(\xi) = \frac{1}{2}(u_1 + u_2) - \frac{1}{2}(u_1 - u_2) \tanh \left( \frac{\xi}{4\nu} \right)(u_1 - u_2) \]

Numerical Solution of Burgers equation

We can solve the equation using our split step spectral method. The equation can be written as
\[ \partial_t u = -\frac{1}{2}\partial_x \left( u^2 \right) + \nu \partial_x^2 u \]
We solve this by solving in Fourier space to give
\[ \partial_t \hat{u} = -\frac{1}{2}ik(u^2) - \nu k^2 \hat{u} \]
Then we solve each of the steps in turn for a small time interval to give
\[ \hat{u}(k, t + \Delta t) = \hat{u}(k, t) - \frac{\Delta t}{2}ik \hat{F} \left( \left| \hat{F}^{-1}(k, t) \right|^2 \right) \]
\[ \hat{u}(k, t + \Delta t) = \hat{u}(k, t + \Delta t) \exp \left( -\nu k^2 \Delta t \right) \]

Exact Solution of Burgers equations

We can find an exact solution to Burgers equation. We want to solve
\[ \partial_t u + uu_x = \nu \partial_x^2 u \]
\[ u(x, 0) = F(x) \]
Frist we write the equation as
\[ \partial_t u + \partial_x \left( \frac{u^2}{2} - \nu \partial_x u \right) = 0 \]
We want to find a function \( \psi(x, t) \) such that
\[ \partial_t \psi = u, \quad \partial_x \psi = \nu \partial_x u - \frac{u^2}{2} \]
Note that because \( \partial_t \partial_x \psi = \partial_t \partial_x \psi \) we will satisfy Burgers equation. This gives us the following equation for \( \psi \)
We introduce the *Cole-Hopf* transformation

\[ \psi = -2\nu \log (\phi) \]

From this we can obtain the three results:

\[ \partial_t \psi = -2\nu \frac{\partial_x \phi}{\phi} \]
\[ \partial^2_x \psi = 2\nu \left( \frac{\partial_x \phi}{\phi} \right)^2 - \frac{2\nu}{\phi} \partial^2_x \phi \]
\[ \partial_t \psi = -2\nu \frac{\partial_t \phi}{\phi} \]

Therefore

\[ \partial_t \psi = \nu \partial^2_x \psi - \frac{1}{2} \left( \partial_x \psi \right)^2 \]

becomes

\[ -2\nu \frac{\partial_t \phi}{\phi} = 2\nu^2 \left( \frac{\partial_x \phi}{\phi} \right)^2 - 2\nu^2 \frac{\partial^2_x \phi}{\phi} - \frac{1}{2} \left( 2\nu \frac{\partial_x \phi}{\phi} \right)^2 \]

or

\[ \partial_t \phi = \nu \partial^2_x \phi \]

which is just the diffusion equation. Note that we also have to transform the boundary conditions. We have

\[ F(x) = u(x, 0) = -2\nu \frac{\partial_t \phi(x, 0)}{\phi(x, 0)} \]

We can write this as

\[ \frac{d}{dx} (\log (\phi)) = -\frac{1}{2\nu} F(x) \]

which has solution

\[ \phi(x, 0) = \Phi(x) = \exp \left( -\frac{1}{2\nu} \int_0^x F(s) \, ds \right) \]

We need to solve

\[ \partial_t \phi \quad \phi(x, 0) = \Phi(x) \]

We take the Fourier transform and obtain

\[ \hat{\partial_t} \hat{\phi} = -k^2 \nu \hat{\phi} \]
\[ \hat{\phi}(k, 0) = \hat{\Phi}(k) \]

which has solution

\[ \hat{\phi}(k, t) = \hat{\Phi}(k) e^{-k^2 \nu t} \]

We can then use the convolution theorem to write

\[ \phi(x, t) = \Phi(x) * F^{-1} \left[ e^{-k^2 \nu t} \right] \]
\[ = \frac{1}{2\sqrt{\pi \nu t}} \int_{-\infty}^{\infty} \Phi(y) \exp \left[ -\frac{(x-y)^2}{4\nu t} \right] \, dy \]

Which can be expressed as

\[ \phi(x, t) = \frac{1}{2\sqrt{\pi \nu t}} \int_{-\infty}^{\infty} \exp \left[ -\frac{f}{2\nu} \right] \, dy \]

where
\[ f(x, y, t) = \frac{1}{2\nu} \int_{0}^{t} F(s) \, ds + \frac{(x - y)^2}{2t} \]

To find \( u \) we recall that

\[
u(x, t) = -2\nu \frac{\partial \phi(x, t)}{\phi(x, t)} = \frac{f_{\infty} \left( \frac{x-y}{t} \right) \exp \left[ -\frac{f}{2\nu} \right] dy}{f_{\infty} \exp \left[ -\frac{f}{2\nu} \right] dy}\]

Category: Nonlinear PDE's Course

- This page was last modified on 13 October 2011, at 18:09.