Connection between KdV and the Schrödinger Equation

From WikiWaves

If we substitute the relationship
\[ \frac{\partial^2}{\partial x^2} w + uw = -\lambda w \]
into the KdV after some manipulation we obtain
\[ \partial_t \lambda w^2 + \partial_x \left( w \partial_x Q - \partial_x w Q \right) = 0 \]
where \( Q = \partial_t w + \partial_x^3 w - 3(\lambda - u) \partial_x w \). If we integrate this equation then we obtain the result that
\[ \partial_t \lambda = 0 \]
provided that the eigenfunction \( w \) is bounded (which is true for the bound state eigenfunctions). This shows that the discrete eigenvalues are unchanged and \( u(x, t) \) evolves according to the KdV. Many other properties can be found.

**Scattering Data**

For the discrete spectrum the eigenfunctions behave like
\[ w_n(x) = c_n(t) e^{-k_n x} \]
as \( x \to \infty \) with
\[ \int_{-\infty}^{\infty} (w_n(x))^2 \, dx = 1 \]
The continuous spectrum looks like
\[
\begin{align*}
  w(x, t) &\approx e^{-ikx} + r(k, t) e^{ikx}, & x \to -\infty \\
  w(x, t) &\approx a(k, t) e^{-ikx}, & x \to \infty
\end{align*}
\]
where \( r \) is the reflection coefficient and \( a \) is the transmission coefficient. This gives us the scattering data at \( t = 0 \)
\[ S(\lambda, 0) = \left\{ \{k_n, c_n(0)\}_{n=1}^{N}, r(k, 0), a(k, 0) \right\} \]
The scattering data evolves as
\[
\begin{align*}
  k_n &= k_n \\
  c_n(t) &= c_n(0) e^{4k_n^3 \lambda t} \\
  r(k, t) &= r(k, 0) e^{8ik^3 \lambda t} \\
  a(k, t) &= a(k, 0)
\end{align*}
\]
We can recover \( u \) from scattering data. We write
Then solve

\[ F (x, t) = \sum_{n=1}^{N} c_n^2 (t) e^{-k_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r (k, t) e^{ikx} dk \]

This is a linear integral equation called the \textit{Gelfand-Levitan-Marchenko} equation. We then find \( u \) from

\[ u(x, t) = 2\partial_x K(x, x, t) \]

**Reflectionless Potential**

In general the IST is difficult to solve. However, there is a simplification we can make when we have a reflectionless potential (which we will see gives rise to the soliton solutions). The reflectionless potential is the case when \( r (k, 0) = 0 \) for all values of \( k \) for some \( u \). In this case

\[ F (x, t) = \sum_{n=1}^{N} c_n^2 (t) e^{-k_n x} \]

then

\[ K (x, t) + \sum_{n=1}^{N} c_n^2 (t) e^{-k_n (x+y)} + \int_{x}^{\infty} K (x, z, t) \sum_{n=1}^{N} c_n^2 (t) e^{-k_n (y+z)} dz = 0 \]

From the equation we can see that

\[ K (x, y, t) = -\sum_{n=1}^{N} v_n (x, t) e^{-k_n y} \]

If we substitute this into the equation,

\[- \sum_{n=1}^{N} v_n (x, t) e^{-k_n y} + \sum_{n=1}^{N} c_n^2 (t) e^{-k_n (x+y)} + \int_{x}^{\infty} - \sum_{m=1}^{N} v_m (x, t) e^{-k_m z} \sum_{n=1}^{N} c_n^2 (t) e^{-k_n (y+z)} dz = 0 \]

which leads to

\[- \sum_{n=1}^{N} v_n (x, t) e^{-k_n y} + \sum_{n=1}^{N} c_n^2 (t) e^{-k_n (x+y)} - \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{c_n^2 (t)}{k_n + k_m} v_m (x) e^{-k_m x} e^{-k_n (y+x)} = 0 \]

and we can eliminate the sum over \( n \) and the \( e^{-k_n y} \) to obtain

\[-v_n (x, t) + \frac{c_n^2 (t)}{k_n + k_m} v_m (x, t) e^{-(k_m + k_n) x} = 0 \]

which is an algebraic (finite dimensional system) for the unknowns \( v_n \).

We can write this as
(I + C) \vec{v} = \vec{f}

where \( f_m = c_m^2(t)e^{-k_m x} \) and the elements of \( C \) are given by

\[
c_{mn} = \frac{c_n^2(t)}{k_n + k_m} e^{-(k_m + k_n)x}
\]

This gives us

\[
K(x, y, t) = -\sum_{m=1}^{N} (I + C)^{-1} \vec{f} e^{-k_m y}
\]

We then find \( u(x, t) \) from \( K \).

**Single Soliton Example**

If \( n = 1 \) (a single soliton solution) we get

\[
K(x, x, t) = -\frac{c_1^2(t)e^{-2k_1x}}{1 + \frac{c_1^2(t)}{e^{-2k_1x}}}
\]

where \( e^{-\alpha} = 1/c_0^2(0) \). Therefore

\[
u(x, t) = 2\partial_x K(x, x, t) = 2k_1 e^{2k_1x - 8k_1^2t - \alpha} \]

where \( \theta = k_1 x - 4k_1^3 t - \alpha/2 \) and \( \sqrt{2k}e^{-\alpha/2} = e^{-k_1 x} \). This is of course the single soliton solution.