Conservation Laws for the KdV

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One of the most interesting features of the KdV is the existence of infinitely many conservation laws. Let's begin with some basics of conservation laws. If we can write our equation of the form

$$\partial_t T(u) + \partial_x X(u) = 0$$

Then we can integrate this equation from $-\infty$ to $\infty$ to obtain

$$\int_{-\infty}^{\infty} \partial_t T(u) \, dx = -\int_{-\infty}^{\infty} \partial_x X(u) \, dx$$

The second integral will be zero if $u \to 0$ as $x \to \pm\infty$. Therefore

$$\partial_t \int_{-\infty}^{\infty} T(u) \, dx = 0$$

so that the quantity

$$\int_{-\infty}^{\infty} T(u) \, dx$$

must be conserved by the solution of the equation. For the KdV we can write

$$\partial_t u + \partial_x \left( 3u^2 + \partial_x^2 u \right) = 0,$$

so that we immediately see that the quantity

$$\int_{-\infty}^{\infty} u \, dx$$

is conserved. This corresponds to conservation of momentum. We can also write the KdV equation as

$$\partial_t \left( u^2 \right) + \partial_x \left( 4u^3 + 2u\partial_x^2 u - (\partial_x u)^2 \right) = 0$$

so that the quantity

$$\int_{-\infty}^{\infty} u^2 \, dx$$
must be conserved. This corresponds to conservation of energy. It turns out that there is an infinite number of conserved quantities and we give here the proof of this.

**Modified KdV**

The modified KdV is

\[
\partial_t v - 3\partial_x \left(v^3\right) + \partial_x^3 v = 0
\]

It is connected to the KdV by Miura's transformation

\[
u = - \left(v^2 + \partial_x v\right)
\]

If we substitute this into the KdV we obtain

\[
\partial_t u + 3\partial_x \left(u^2\right) + \partial_x^3 u = -(2v + \partial_x) \left(\partial_t v - 3\partial_x \left(v^3\right) + \partial_x^3 v\right)
\]

Note that this shows that every solution of the mKdV is a solution of the KdV but not vice versa.

**Proof of an Infinite Number of Conservation Laws**

An ingenious proof of the existence of an infinite number of conservation laws can be obtain from a generalization of Miura's transformation

\[
u = w - \varepsilon \partial_x w - \varepsilon^2 w^2
\]

If we substitute this into the KdV we obtain

\[
\partial_t u + 3\partial_x \left(u^2\right) + \partial_x^3 u = \left(1 - \varepsilon \partial_x - 2\varepsilon^2 w\right) \left(\partial_t w + 6 \left(w - \varepsilon^2 w^2\right) \partial_x w + \partial_x^3 w\right)
\]

Therefore \( u \) solves the KdV equation provided that

\[
\partial_t w + 6 \left(w - \varepsilon^2 w^2\right) \partial_x w + \partial_x^3 w = 0
\]

We write the solution to this equation as a formal power series

\[
w \left(x, t, \varepsilon\right) = \sum_{n=0}^{\infty} \varepsilon^n w_n \left(x, t\right)
\]

Since the equation is in conservation form then

\[
\int_{-\infty}^{\infty} w \left(x, t, \varepsilon\right) \, dx = \text{constant}
\]

and since this is true for all \( \varepsilon \) this implies that
\[ \int_{-\infty}^{\infty} w_n(x, t) \, dx = \text{constant} \]

We then consider the expression

\[ u = w - \varepsilon \partial_x w - \varepsilon^2 w^2 \]

which implies that

\[ u = \sum_{n=0}^{\infty} \varepsilon^n w_n(x, t) - \varepsilon \partial_x \left( \sum_{n=0}^{\infty} \varepsilon^n w_n(x, t) \right) - \varepsilon^2 \left( \sum_{n=0}^{\infty} \varepsilon^n w_n(x, t) \right)^2 \]

If we equate powers of \( \varepsilon \) we obtain

\[
\begin{align*}
0 &= w_0 \\
0 &= w_1 - \partial_x w_0 \\
0 &= w_2 - \partial_x w_1 - w_0^2 \\
0 &= w_3 - \partial_x w_2 - 2w_0 w_1
\end{align*}
\]

We can solve recursively to obtain

\[
\begin{align*}
w_0 &= u \\
w_1 &= \partial_x u \\
w_2 &= \partial_x^2 u + u^2 \\
w_3 &= \partial_x^3 u + 4u \partial_x u
\end{align*}
\]

Note that each of the odd conservation laws \( (w_1, w_3 \text{ etc.}) \) are just the derivative (with some modification) of the previous law and therefore does not actually provide a new conservation law.


Category: Nonlinear PDE's Course

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