1. Consider the following system of equations

\[
\frac{da}{dt} = -kab \\
\frac{db}{dt} = kab \\
a(0) = a_0, \quad b(0) = b_0
\]

(a) Solve these equations exactly. [6 marks]

(b) Draw the solutions for different values of the parameters using MATLAB. Hand in the plots and your MATLAB code. [4 marks]

2. The discrete Fourier transform (DFT) is given by

\[
\hat{c}_m = \sum_{n=0}^{N-1} c_n e^{-2\pi i mn/N} \quad \text{for} \quad m = 0, \ldots, N-1
\]

for a vector \( c \) of length \( N \). The inverse discrete Fourier transform (IDFT) is given by

\[
c_n = \frac{1}{N} \sum_{m=0}^{N-1} \hat{c}_m e^{2\pi i mn/N} \quad \text{for} \quad n = 0, \ldots, N-1.
\]

(a) Program these in MATLAB. Apply this to the function \( f(x) = \exp(-x^2) \) with \( x = \{-1, -0.8, \ldots, 1\} \). Hand in your values for \( \hat{c}_m \) and \( c_n \) and show that they agree with the FFT and IFFT commands in MATLAB. [4 marks]

(b) Prove that if we apply the DFT followed by the IDFT we obtain identity operator (matrix). [4 marks]
3. Solve the reaction diffusion equations

\[
\begin{align*}
\partial_t \alpha &= \partial_z^2 \alpha - \alpha \beta \\
\partial_t \beta &= \partial_z^2 \beta + \alpha \beta
\end{align*}
\]

numerically for the following initial conditions. Hand in a plot of the solution for various times and your MATLAB code.

(a) \[
\begin{align*}
\alpha(z, 0) &= 1, \\
\beta(z, 0) &= \frac{e^{-5z^2}}{10}
\end{align*}
\]

(b) \[
\begin{align*}
\alpha(z, 0) &= e^{-z^2}, \\
\beta(z, 0) &= \frac{e^{-10z^2}}{10}
\end{align*}
\]

4. Burgers equation is

\[
\partial_t u + u \partial_x u = \nu \partial_x^2 u
\]

We can find a travelling wave solution by assuming that

\[
u (x, t) = u(x - ct) = u(\xi)
\]

This leads to the equations

\[-cu' + u'u - \nu u'' = 0 \quad (1)
\]

The phase plane for this system, writing \(w = u'\) is

\[
\begin{align*}
\frac{du}{d\xi} &= w \\
\frac{dw}{d\xi} &= \frac{1}{\nu} (w (u - c))
\end{align*}
\]

(a) Draw the phase plane and show typical solutions on this (you may assume that \(\nu = 1 = c\)).

(b) Solve equation (1) exactly to obtain

\[
u (\xi) = \frac{1}{2} (u_1 + u_2) - \frac{1}{2} (u_1 - u_2) \tanh \left[\left(\frac{\xi}{4\nu}\right) (u_1 - u_2)\right]
\]

where you can assume \(u_1 < u < u_2\) and also you can assume \(u(0) = (u_1 + u_2)/2\).

(c) Solve the system of equations numerically and show that you obtain the same solution as given in b). Hand in plots and the MATLAB code.
5. Solve Burgers equation

\[ \partial_t u + u \partial_x u = \frac{1}{10} \partial_x^2 u \]

numerically using a split step spectral method for the following initial conditions

(a) \[ u (x, 0) = \exp (-x^2) \]

(b) \[ u (x, 0) = \begin{cases} 1, & x \in [-3, 3] \\ 0, & x \notin [-3, 3] \end{cases} \]

[3 marks]

[3 marks]

Hand in plots and the MATLAB code.

6. (a) Show that the function

\[ \phi (x, t) = 1 + \sqrt{\frac{a}{t}} e^{-x^2/4\nu t} \]

with positive constant \(a\) satisfies the dispersion equation

\[ \frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2} \]

[4 marks]

(b) Use the Cole-Hopf transformation to find the corresponding solution \(u(x, t)\) to Burgers equation

\[ \partial_t u + u \partial_x u = \nu \partial_x^2 u \]

[3 marks]

(c) Show that this solution is antisymmetric and that the area under the solution for \(x > 0\) \(A(t)\) is given by

\[ A(t) = 2\nu \log \left( 1 + \sqrt{\frac{a}{t}} \right) \]

[3 marks]

TotalMarks: 50