CHAPTER 1

Homogeneous Spaces

Let $G$ be a compact topological group and $K$ be a compact Hausdorff space. We say that $G$ acts transitively on $K$ if there is a continuous map $G \times K \to K: (g, k) \mapsto g(k)$ such that

(i) $e(k) = k$ for all $k \in K$ ($e$ is the identity of $G$);
(ii) $(g_1 g_2)(k) = g_1(g_2(k))$ for all $g_1, g_2 \in G, h \in K$;
(iii) given $k_1, k_2 \in K$ there is a $g \in G$ so that $g(k_1) = k_2$.

It is noteworthy that each $g \in G$ may be viewed as a homeomorphism of $K$ onto itself; after all, the map $k \mapsto g(k)$ is continuous and has $k \mapsto g^{-1}(k)$ as an inverse.

Condition (iii) says, in particular, that the space $K$ is homogeneous i.e., we can move points of $K$ around $K$ via homeomorphisms (members of $G$, in fact) of $K$ onto itself.

If $\mu$ is the unique translation invariant Borel probability on $G$ then $\mu$ induces a $G$–invariant Borel probability on $K$. This is an important construction, one worth understanding in general as well as in special cases.

Suppose $H$ is a closed subgroup of the compact topological group $G$. Consider the set $G/H$ with the so-called ‘quotient topology,’ that is, the strongest topology that makes the natural map $q_H: G \to G/H$ (taking $g \in G$ to $gH \in G/H$) continuous; so $U \subseteq G/H$ is open precisely when $q_H^{-1}(U)$ is open in $G$. In other words, a typical open set in $G/H$ is of the form $\{xH : x \in V\}$ when $V$ is open in $G$. Because $H$ is supposed to be closed, this topology is Hausdorff; because $q_H$ is continuous and surjective, $G/H$ is compact.

More is so. $G$ acts transitively on $G/H$. The map $(g, g'H) \mapsto gg'H$ fits the bill in the definition.

In fact, any transitive action of a compact group on a compact space is of the sort just described. To be sure we need to tell when seemingly different spaces are the same under $G$’s action. Let $G$ act transitively on each of the compact Hausdorff spaces $K_1, K_2$. We say that $K_1$ and $K_2$ are isomorphic under $G$’s action if there is a homeomorphism $\phi: K_1 \to K_2$ such that

$$\phi(g(k_1)) = g(\phi(k_1))$$

for each $k_1 \in K_1$.

**Theorem 1.1 (Weil).** Let the compact group $G$ act transitively on the compact Hausdorff space $K$. Then there is a closed subgroup $H$ of $G$ such that $K$ and $G/H$ are isomorphic under $G$’s action.

**Proof.** Fix $k_0 \in K$. Look at

$$H = \{g \in G : g(k_0) = k_0\}.$$
$H$ is called the **isotopy subgroup**. It is plain that $H$ is a closed subgroup of $G$. A natural candidate for the isomorphism of $G/H$ and $K$ is at hand: $\phi : G/H \to K$ given by

$$\phi(gH) = g(k_0).$$

For $g_1, g_2 \in G$, $g_1(k_0) = g_2(k_0)$ precisely when

$$g_1^{-1}(g_2(k_0)) = g_1^{-1}(g_1(k_0)) = e(k_0) = k_0,$$

or $g_1^{-1}g_2 \in H$, which is tantamount to $g_1H = g_2H$. This assures us that $\phi$ is well-defined and injective.

The transitivity of $G$’s action ensures $\phi$’s surjectivity. To see that $\phi$ is also continuous, fix $g \in G$ and let $V$ be an open set in $K$ containing $g(k_0)$. By the continuity of the map $(g, k) \to g(k)$ on $G \times K$, there is an open set $U$ in $G$ which contains $g$ so that $u(k_0) \in V$ for all $u \in U$. But $q_H(U)$ is open in $G/H$’s quotient topology and $q_H(U) \subseteq \phi^{-1}(V)$. This shows that $\phi$ is a continuous bijection between the compact Hausdorff spaces $G/H$ and $K$; as such $\phi$ is a homeomorphism.

Further if $g_1, g_2 \in G$ then

$$g_1(\phi(g_2H)) = g_1(g_2(k_0)) = (g_1g_2)(k_0) = \phi(g_1(g_2H)).$$

Thus $G/H$ and $K$ are isomorphic under $G$’s action. \hfill $\square$

Note that because of this isomorphism theorem we can consider any $G/H$ where $H$ is the isotopy subgroup associated with any $k_0 \in K$.

Now we’re ready for the main course.

**Theorem 1.2 (Weil).** Suppose the compact group $G$ acts transitively on the compact Hausdorff space $K$. Then there is a unique $G$–invariant regular Borel probability measure on $K$.

**Proof.** We identify $K$ with the isotopy subgroup $G/H$ as in our previous theorem. Let

$$q_H : G \to G/H$$

be the natural quotient map. Suppose $\mu$ is the normalized Haar measure on $G$ and define $\mu_{G/H}$ on $G/H$ by

$$\mu_{G/H}(B) = \mu(q_H^{-1}(B))$$

for any Borel set $B \subseteq G/H$.

If $g \in G$ and $B$ is Borel subset of $G/H$ then

$$g(q_H^{-1}(B)) = \{gx : xH \in B\}$$

$$= \{gx : gH \in gB\}$$

$$= q_H^{-1}(gB).$$
Therefore
\[
\mu_{G/H}(gB) = \mu(q^{-1}_H(gB)) = \mu(g(q^{-1}_H(B))) = \mu(q^{-1}_H(B)) = \mu_{G/H}(B),
\]
and \(\mu_{G/H}\) is \(G\)-invariant.

Uniqueness is a touchier issue, as is always the case it seems. We take a close look at how regular countably additive Borel measures, members of \(rca(\mathcal{B}_0(G)) = C(K)^*\) act on \(C(G)\).

Take \(\phi \in C(G)\) and \(g \in G\). Define \(\phi_g \in C(G)\) by
\[
\phi_g(x) = \phi(gx).
\]
Denote by \(\mu_H\) the Haar measure (normalized so \(\mu_H = 1\)) on \(H\). The map \(G \rightarrow C(G)\) that takes \(g\) to \(\phi_g\) is uniformly continuous (this is an easy modification of Theorem ??) so that
\[
\hat{\phi}(g) = \int_H \phi_g(h) d\mu_H(h), \quad g \in G
\]
defines a member \(\hat{\phi}\) of \(C(G)\).

Suppose \(g_1H = g_2H\). Then \(g_1^{-1}g_2 \in H\),
\[
\hat{\phi}(g_1) = \int_H \phi_{g_1}(h) d\mu_H(h)
\]
\[
= \int_H \phi_{g_1}(g_1^{-1}g_2h) d\mu_H(h) \quad \text{(by } \mu_H\text{'s invariance and } g_1^{-1}g_2 \in H)\]
\[
= \int_H \phi(g_2H) d\mu_H(h)
\]
\[
= \int_H \phi_{g_2}(h) d\mu_H(h) = \hat{\phi}(g_2).
\]
Therefore \(\hat{\phi}\) is constant on the left cosets of \(H\) so we can lift \(\hat{\phi}\) to a continuous function \(\hat{\phi}\) on \(G/H\):
\[
\hat{\phi}(gH) = \hat{\phi}(g).
\]
To summarize: if \(\phi \in C(G)\) then we define \(\hat{\phi} \in C(G)\) and from this we get \(\tilde{\phi} \in C(G/H)\).

Remarkably, each member of \(C(G/H)\) comes about from this procedure. In fact, if \(f \in C(G/H)\)
then \( f \circ q_H \in C(G) \) and for any \( g \in G \)
\[
(f \circ q_H)(gH) = \int_H (f \circ q_H)(gh) \, d\mu_H(h) = \int_H f(gh)d\mu_H(h) = f(gH)\mu_H(H) = f(gH).
\]

In other words, \( f = (f \circ \widetilde{q}_H) \).

Now we look at \( G \)'s action. Take any \( G \)-invariant regular Borel probability measure \( \nu \) on \( G/H \). For \( \phi \in C(G) \) define
\[
\lambda(\phi) = \int_{G/H} \hat{\phi}(gH) \, d\nu(gH).
\]
Then \( \lambda \) is a probability measure in \( C(G)^\ast \). Moreover, \( \lambda \) is translation invariant. Indeed if \( x \in G \)
\[
\lambda(\phi_x) = \int_{G/H} \hat{\phi}_x(gH) \, d\nu(gH)
= \int_{G/H} \hat{\phi}_x(g) \, d\nu(gH)
= \int_{G/H} \phi(xg) \, d\nu(gH)
= \int_{G/H} \hat{\phi}(xgH) \, d\nu(gH)
= \int_{G/H} \hat{\phi}(gH) \, d\nu(gH) = \lambda(\phi).
\]
So \( \lambda \) is nothing else but normalized Haar measure on \( G \).

If \( \nu_1 \) and \( \nu_2 \) are \( G \)-invariant regular Borel probabilities on \( G/H \) and if \( x = (x \circ q_H) \in C(G/H) \) then
\[
\nu_1(x) = \int_{G/H} x(gH) \, d\nu_1(gH)
= \int_{G/H} (x \circ q_H)(gH) \, d\nu_1(gH)
= \lambda(x \circ q_H)
= \int_{G/H} (x \circ q_H)(gH) \, d\nu_2(gH)
= \nu_2(x);
\]
1. HOMOGENEOUS SPACES

in other words, \( \nu_1 \) and \( \nu_2 \) are the same.

The worth of an abstract construction lies, at least in part, in its applicability to concrete cases. Our first application is classical and was well-known before Weil’s general theorem. It is, nonetheless, interesting and important.

Our setting: \( O(n) \), the orthogonal group of order \( n \) is our compact group; \( S^{n-1} \), the unit sphere in \( \mathbb{R}^n \) is our compact Hausdorff space. The action of \( O(n) \) on \( S^{n-1} \) is given, naturally by

\[(u, x) \to u(x).\]

It is easy to verify that \( O(n) \) acts on \( S^{n-1} \) in a suitable fashion! Transitivity follows by letting \( x_1, x'_1 \in S^{n-1} \); choose orthonormal bases \( x_1, x_2, \ldots, x_n \) and \( x'_1, x'_2, \ldots, x'_n \) for \( \mathbb{R}^n \), and let \( u : \mathbb{R}^n \to \mathbb{R}^n \) be the member of \( O(n) \) that takes \( x_j \) to \( x'_j \) for \( j = 1, \ldots, n \).

Acknowledging the descriptions of members of \( O(n) \) as rotations of \( \mathbb{R}^n \), a direct application of Weil’s theorem says: There is a unique rotation-invariant regular Borel probability measure on \( S^{n-1} \).

Geometry is replete with examples of compact Hausdorff spaces that are homogeneous spaces on which various compact groups act transitively.

Here are a few more.

Again our group is \( O(n) \). This time our underlying compact space is

\[ \sum(n) = \{(x, y) \in S^{n-1} \times S^{n-1} : x \perp y \}, \]

where \( x \perp y \) means \( x \) is perpendicular to \( y \). Note that \( (x, y) \in \sum(n) \) precisely when for any real valued numbers \( a, b \):

\[ ||ax + by||^2 = a^2 + b^2. \]

It is easy to see from this that \( \sum(n) \) is a closed subset of \( S^{n-1} \times S^{n-1} \), hence, is compact. The action of \( O(n) \) is natural enough, too: \((u, (x, y)) \to (ux, uy)\). It is quick and reasonably painless to see that \( O(n) \) acts transitively on \( \sum(n) \).

One more. Let \( 1 \leq m \leq n \). Denote by \( \sum^{(m)}(n) \) the set

\[ \sum^{(m)}(n) = \left\{(x_1, \ldots, x_m) \in S^{n-1} \times \cdots \times S^{n-1} : \{x_1, \ldots, x_m\} \text{ is orthonormal}\right\}. \]

Note that \( (x_1, \ldots, x_m) \in \sum^{(m)}(n) \) precisely when regardless of the real numbers \( a_1, \ldots, a_m \), we have

\[ \left\| \sum_{j=1}^{m} a_j x_j \right\|^2 = \sum_{j=1}^{m} a_j^2. \]

This in mind, \( \sum^m(n) \) is a compact set of \( (S^{n-1})^m \) is easy to see; moreover, the action of \( O(n) \) on \( \sum^{(m)}(n) \) is given by

\[ (u, (x_1, \ldots, x_m)) \to (ux_1, ux_2, \ldots, ux_m) \]
is a transitive one, establishing, with a modicum of tender love and care, that \( O(n) \) acts transitively on \( \sum^{(m)}(n) \).

Next let \( G_m(n) \) denote the \( m \)--dimensional Grassmanian manifold, that is, \( G_m(n) \) is the space of all \( m \)--dimensional linear subspaces of \( \mathbb{R}^n \). There is a natural surjection of \( \sum^{(m)}(n) \) onto \( G_m(n) \) that takes \( (x_1, \ldots, x_m) \in \sum^{(m)}(n) \) to the linear span of \( \{x_1, \ldots, x_m\} \in G_m(n) \). If we equip \( G_m(n) \) with the natural quotient topology the result is a compact Hausdorff space. Clearly \( O(n) \) acts transitively on \( G_m(n) \). The map reflecting the action of \( O(n) \) on \( G_m \) is plain: if \( \{x_1, \ldots, x_m\} \) is an orthonormal set in \( \mathbb{R}^n \) then

\[
(u, \text{span}\{x_1, \ldots, x_m\}) = \text{span}\{ux_1, \ldots, ux_m\}.
\]

Here we interject that the geometry imparted above on \( G_m(n) \) is such that if \( E = \text{span}\{x_1, \ldots, x_m\} \) and \( E' = \text{span}\{x'_1, \ldots, x'_m\} \) are members of \( G_m(n) \) and if each \( x_k \) is close to \( x'_k \) in \( \mathbb{R}^n \) then \( E \) is close to \( E' \) in \( G_m(n) \).

In this way we find that there is a unique rotation invariant probability Borel measure on the \( n \)--dimensional Grassmanian manifold \( G_m(n) \).