CHAPTER 1

Homogeneous Spaces

Let G be a compact topological group and K be a compact Hausdorff space. We say that G **acts transitively** on K if there is a continuous map $G \times K \to K : (g, k) \mapsto g(k)$ such that

(i) $e(k) = k$ for all $k \in K$ (e is the identity of G);
(ii) $(g_1 g_2)(k) = g_1(g_2(k))$ for all $g_1, g_2 \in G, h \in K$;
(iii) given $k_1, k_2 \in K$ there is a $g \in G$ so that $g(k_1) = k_2$.

It is noteworthy that each $g \in G$ may be viewed as a homeomorphism of K onto itself; after all, the map $k \to g(k)$ is continuous and has $k \to g^{-1}(k)$ as an inverse.

Condition (iii) says, in particular, that the space K is *homogeneous* i.e., we can move points of K around K via homeomorphisms (members of G, in fact) of K onto itself.

If μ is the unique translation invariant Borel probability on G then μ induces a $G-$invariant Borel probability on K. This is an important construction, one worth understanding in general as well as in special cases.

Suppose H is a closed subgroup of the compact topological group G. Consider the set G/H with the so-called 'quotient topology,' that is, the strongest topology that makes the natural map $q_H : G \to G/H$ (taking $g \in G$ to $gH \in G/H$) continuous; so $U \subseteq G/H$ is open precisely when $\overleftarrow{q_H}(U)$ is open in G. In other words, a typical open set in G/H is of the form $\{xH : x \in V\}$ when V is open in G. Because H is supposed to be closed, this topology is Hausdorff; because q_H is continuous and surjective, G/H is compact.

More is so. G acts transitively on G/H. The map $(g, g^{'}H) \to gg^{'}H$ fits the bill in the definition.

In fact, any transitive action of a compact group on a compact space is of the sort just described. To be sure we need to tell when seemingly different spaces are the same under G's action. Let G act transitively on each of the compact Hausdorff spaces K_1, K_2. We say that K_1 **and** K_2 **are isomorphic under** G**'s action** if there is a homeomorphism $\phi : K_1 \to K_2$ such that

$$\phi(g(k_1)) = g(\phi(k_1))$$

for each $k_1 \in K_1$.

THEOREM 1.1 (Weil). *Let the compact group G act transitively on the compact Hausdorff space K. Then there is a closed subgroup H of G such that K and G/H are isomorphic under G's action.*

PROOF. Fix $k_0 \in K$. Look at

$$H = \{g \in G : g(k_0) = k_0\}.$$

H is called the **isotopy subgroup**. It is plain that H is a closed subgroup of G. A natural candidate for the isomorphism of G/H and K is at hand: $\phi : G/H \to K$ given by

$$\phi(gH) = g(k_0).$$

For $g_1, g_2 \in G$, $g_1(k_0) = g_2(k_0)$ precisely when

$$g_1^{-1}(g_2(k_0)) = g_1^{-1}(g_1(k_0)) = e(k_0) = k_0,$$

or $g_1^{-1}g_2 \in H$, which is tantamount to $g_1H = g_2H$. This assures us that ϕ is well-defined and injective.

The transitivity of G's action ensures ϕ's surjectivity. To see that ϕ is also continuous, fix $g \in G$ and let V be an open set in K containing $g(k_0)$. By the continuity of the map

$$(g, k) \to g(k)$$

on $G \times K$, there is an open set U in G which contains g so that $u(k_0) \in V$ for all $u \in U$. But $q_H(U)$ is open in G/H's quotient topology and $q_H(U) \subseteq \phi^{\leftarrow}(V)$. This shows that ϕ is a continuous bijection between the compact Hausdorff spaces G/H and K; as such ϕ is a homeomorphism.

Further if $g_1, g_2 \in G$ then

$$g_1(\phi(g_2H)) = g_1(g_2(k_0)) = (g_1g_2)(k_0) = \phi(g_1(g_2H)).$$

Thus G/H and K are isomorphic under G's action. □

Note that because of this isomorphism theorem we can consider *any* G/H where H is the isotopy subgroup associated with *any* $k_0 \in K$.

Now we're ready for the main course.

Theorem 1.2 (Weil). *Suppose the compact group G acts transitively on the compact Hausdorff space K. Then there is a unique $G-$invariant regular Borel probability measure on K.*

Proof. We identify K with the isotopy subgroup G/H as in our previous theorem. Let

$$q_H : G \to G/H$$

be the natural quotient map. Suppose μ is the normalized Haar measure on G and define $\mu_{G/H}$ on G/H by

$$\mu_{G/H}(B) = \mu(q_H^{\leftarrow}(B))$$

for any Borel set $B \subseteq G/H$.

If $g \in G$ and B is Borel subset of G/H then

$$\begin{aligned} g(q_H^{\leftarrow}(B)) &= \{gx : xH \in B\} \\ &= \{gx : gxH \in gB\} \\ &= q_H^{\leftarrow}(gB). \end{aligned}$$

Therefore

$$\begin{aligned}\mu_{G/H}(gB) &= \mu(q_H^{\leftarrow}(gB)) \\ &= \mu(g(q_H^{\leftarrow}(B)) \\ &= \mu(q_H^{\leftarrow}(B)) = \mu_{G/H}(B),\end{aligned}$$

and $\mu_{G/H}$ is $G-$invariant.

Uniqueness is a touchier issue, as is always the case it seems. We take a close look at how regular countably additive Borel measures, members of $\text{rca}(\mathcal{B}_o(G)) = C(K)^*$ act on $C(G)$.

Take $\phi \in C(G)$ and $g \in G$. Define $\phi_g \in C(G)$ by

$$\phi_g(x) = \phi(gx).$$

Denote by μ_H the Haar measure (normalized so $\mu_H = 1$) on H. The map $G \to C(G)$ that takes g to ϕ_g is uniformly continuous (this is an easy modification of Theorem **??**) so that

$$\hat{\phi}(g) = \int_H \phi_g(h) d\mu_H(h), \;\; g \in G$$

defines a member $\hat{\phi}$ of $C(G)$.

Suppose $g_1H = g_2H$. Then $g_1^{-1}g_2 \in H$,

$$\begin{aligned}\hat{\phi}(g_1) &= \int_H \phi_{g_1}(h)\, d\mu_H(h) \\ &= \int_H \phi_{g_1}(g_1^{-1}g_2h)\, d\mu_H(h) \;\; (\text{by } \;\mu_H\text{'s invariance and } \; g_1^{-1}g_2 \in H) \\ &= \int_H \phi(g_2H)\, d\mu_H(h) \\ &= \int_H \phi_{g_2}(h)\, d\mu_H(h) = \hat{\phi}(g_2).\end{aligned}$$

Therefore $\hat{\phi}$ is constant on the left cosets of H so we can lift $\hat{\phi}$ to a continuous function $\tilde{\phi}$ on G/H:

$$\tilde{\phi}(gH) = \hat{\phi}(g).$$

To summarize: if $\phi \in C(G)$ then we define $\hat{\phi} \in C(G)$ and from this we get $\tilde{\phi} \in C(G/H)$.

Remarkably, each member of $C(G/H)$ comes about from this procedure. In fact, if $f \in C(G/H)$

then $f \circ q_H \in C(G)$ and for any $g \in G$

$$\begin{aligned}
(\widetilde{f \circ q_H})(gH) &= (\widehat{f \circ q_H})(g) \\
&= \int_H (f \circ q_H)_g(h)\, d\mu_H(h) \\
&= \int_H (f \circ q_H)(gh) d\mu_H(h) \\
&= \int_H f(ghH) d\mu_H(h) \\
&= \int_H f(gH) d\mu_H(h) \\
&= f(gH)\mu_H(H) = f(gH).
\end{aligned}$$

In other words, $f = (\widetilde{f \circ q_H})$.

Now we look at G's action. Take *any* $G-$invariant regular Borel probability measure ν on G/H. For $\phi \in C(G)$ define

$$\lambda(\phi) = \int_{G/H} \tilde{\phi}(gH)\, d\nu(gH).$$

Then λ is a probability measure in $C(G)^\star$. Moreover, λ is translation invariant. Indeed if $x \in G$

$$\begin{aligned}
\lambda(\phi_x) &= \int_{G/H} \tilde{\phi}_x(gH)\, d\nu(gH) \\
&= \int_{G/H} \hat{\phi}_x(g)\, d\nu(gH) \\
&= \int_{G/H} \phi(xg)\, d\nu(gH) \\
&= \int_{G/H} \tilde{\phi}(xgH)\, d\nu(gH) \\
&= \int_{G/H} \tilde{\phi}(gH)\, d\nu(gH) = \lambda(\phi).
\end{aligned}$$

So λ is nothing else but normalized Haar measure on G.

If ν_1 and ν_2 are $G-$invariant regular Borel probabilities on G/H and if $x = (\widetilde{x \circ q_H}) \in C(G/H)$ then

$$\begin{aligned}
\nu_1(x) &= \int_{G/H} x(gH)\, d\nu_1(gH) \\
&= \int_{G/H} (\widetilde{x \circ q_H})(gH)\, d\nu_1(gH) \\
&= \lambda(x \circ q_H) \\
&= \int_{G/H} (\widetilde{x \circ q_H})(gH)\, d\nu_2(gH) \\
&= \nu_2(x);
\end{aligned}$$

in other words, ν_1 and ν_2 are the same. $\square$

The worth of an abstract construction lies, at least in part, it its applicability to concrete cases. Our first application is classical and was well-known before Weil's general theorem. It is, nonetheless, interesting and important.

Our setting: $O(n)$, the orthogonal group of order n is our compact group; S^{n-1}, the unit sphere in $\mathbb{R}^n$ is our compact Hausdorff space. The action of $O(n)$ on S^{n-1} is given, naturally by

$$(u, x) \to u(x).$$

It is easy to verify that $O(n)$ acts on S^{n-1} in a suitable fashion! Transitivity follows by letting $x_1, x_1^{'} \in S^{n-1}$; choose orthonormal bases $x_1, x_2, \ldots, x_n$ and $x_1^{'}, x_2^{'}, \ldots, x_n^{'}$ for $\mathbb{R}^n$, and let $u : \mathbb{R}^n \to \mathbb{R}^n$ be the member of $O(n)$ that takes x_j to $x_j^{'}$ for $j = 1, \ldots, n$.

Acknowledging the descriptions of members of $O(n)$ as rotations of $\mathbb{R}^n$, a direct application of Weil's theorem says: *There is a unique rotation-invariant regular Borel probability measure on* S^{n-1}.

Geometry is replete with examples of compact Hausdorff spaces that are homogeneous spaces on which various compact groups act transitively.

Here are a few more.

Again our group is $O(n)$. This time our underlying compact space is

$$\sum(n) = \{(x, y) \in S^{n-1} \times S^{n-1} : x \perp y\},$$

where $x \perp y$ means x is perpendicular to y. Note that $(x, y) \in \sum(n)$ precisely when for any real valued numbers a, b:

$$||ax + by||^2 = a^2 + b^2.$$

It is easy to see from this that $\sum(n)$ is a closed subset of $S^{n-1} \times S^{n-1}$, hence, is compact. The action of $O(n)$ is natural enough, too: $(u, (x, y)) \to (ux, uy)$. It is quick and reasonably painless to see that $O(n)$ acts transitively on $\sum(n)$.

One more. Let $1 \le m \le n$. Denote by $\sum^{(m)}(n)$ the set

$$\sum^{(m)}(n) = \left\{ (x_1, \ldots, x_m) \in \underbrace{S^{n-1} \times \cdots \times S^{n-1}}_{m \text{ times}} : \{x_1, \ldots, x_m\} \text{ is orthonormal} \right\}.$$

Note that $(x_1, \ldots, x_m) \in \sum^{(m)}(n)$ precisely when regardless of the real numbers $a_1, \ldots a_m$, we have

$$\left|\left| \sum_{j=1}^{m} a_j x_j \right|\right|^2 = \sum_{j=1}^{m} a_j^2.$$

This in mind, $\sum^m(n)$ is a compact set of $(S^{n-1})^m$ is easy to see; moreover, the action of $O(n)$ on $\sum^{(m)}(n)$ is given by

$$(u, (x_1, \ldots, x_m)) \to (ux_1, ux_2, \ldots ux_m)$$

is a transitive one, establishing, with a modicum of tender love and care, that $O(n)$ acts transitively on $\sum^{(m)}(n)$.

Next let $\mathcal{G}_m(n)$ denote the $m-$dimensional Grassmanian manifold, that is, $\mathcal{G}_m(n)$ is the space of all $m-$dimensional linear subspaces of $\mathbb{R}^n$. There is a natural surjection of $\sum_{(m)}(n)$ onto $\mathcal{G}_m(n)$ that takes $(x_1, \ldots, x_m) \in \sum^{(m)}(n)$ to the linear span of $\{x_1, \ldots, x_m\} \in \mathcal{G}_m(n)$. If we equip $\mathcal{G}_m(n)$ with the natural quotient topology the result is a compact Hausdorff space. Clearly $O(n)$ acts transitively on $\mathcal{G}_m(n)$. The map reflecting the action of $O(n)$ on $\mathcal{G}_m$ is plain: if $\{x_1, \ldots, x_m\}$ is an orthonormal set in $\mathbb{R}^n$ then

$$(u, \mathrm{span}\{x_1, \ldots, x_m\}) = \mathrm{span}\{ux_1, \ldots, ux_m\}.$$

Here we interject that the geometry imparted above on $\mathcal{G}_m(n)$ is such that if $E = \mathrm{span}\{x_1, \ldots, x_m\}$ and $E^{'} = \mathrm{span}\{x_1^{'}, \ldots, x_m^{'}\}$ are members of $\mathcal{G}_m(n)$ and if each x_k is close to $x_k^{'}$ in $\mathbb{R}^n$ then E is close to $E^{'}$ in $\mathcal{G}_m(n)$.

In this way we find that there is a unique rotation invariant probability Borel measure on the $n-$dimensional Grassmanian manifold $\mathcal{G}_m(n)$.