Amenability, measure and randomness in groups

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Outline

Three themes of 19^{th} century mathematics

J. Fourier (1807) – diffusion of heat. Fourier analysis,
$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

E. Galois (1830) – unsolvability of quintic equations. Finite group theory, simple groups.

F. Klein (1872) – geometry from symmetry groups.
S. Lie, transformation groups (1888–1893).

At the turn of the 20^{th} Century – analysis

Theory of integration (B. Riemann, H. Lebesgue).

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x,\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x.$$

Which functions can be integrated? Convolution.

For which values of x does the Fourier series converge to $f(x)$? How to describe 'exceptional sets' where the series does not converge? Set theory (G. Cantor). The Axiom of Choice.

Function spaces, approximation and convergence in infinite-dimensional spaces.

At the turn of the 20^{th} Century – finite groups

Jordan-Hölder Theorem, Sylow theorems.
New examples of sporadic simple groups and families of groups of Lie type.

Linear representations of finite groups.
The group algebra, $\mathbb{C}G = \{\sum_{x \in G} \lambda_x x \mid \lambda_x \in \mathbb{C}\}$.
Characters and Maschke's Theorem.

F. Frobenius, L. Dickson, W. Burnside.

At the turn of the 20th Century – geometry

D. Hilbert. Foundations of geometry.
Third problem – are all polyhedra scissors congruent? *No*
Fifth problem – are continuous groups analytic? *Yes*

H. Poincaré. Fundamental group of a topological space.
Geometric group theory.

Size and algebras for infinite groups

How to extend techniques for finite groups to infinite groups?
Functions can be averaged over the group by restricting the functions considered.

Measure the size of a subset of G by counting its elements.
- typically infinite, not much use.

Group algebra: $\ell^1(G) = \{\sum_{x \in G} \lambda_x x \mid \sum_{x \in G} |\lambda_x| < \infty\}$.
- ensures that the series $\sum_{y \in G} \lambda_{xy^{-1}} \mu_y$ converges.

Area and volume

Area in the plane.

The area of any polygon, P, may be found because:
- area is finitely additive;
- area is invariant under euclidean motions; and
- P is scissors congruent to a square.

Finding the area of other regions requires *countable additivity*.
- if $R = \bigcup_{n\in\mathbb{N}} R_n$ where $R_m \cap R_n = \emptyset$ if $m \neq n$, then

$$\mathrm{Area}(R) = \sum_{n\in\mathbb{N}} \mathrm{Area}(R_n).$$

Measure theory and σ-algebras. Borel and measurable sets. Not every subset of $\mathbb{R}$ is measurable (AC).

Additional structures on infinite groups

Theorem (A. Haar)

For each locally compact group G there is a positive, regular measure, m, on the σ-algebra of Borel subsets of G that is invariant under left translation. This measure is unique up to a scalar multiple.

Theorem (A. Weil)

Let G be a group for which there is a left-invariant positive measure on a σ-algebra of subsets of G. Then G is measure isomorphic to a locally compact group and its Haar measure.

The solution to Hilbert's Fifth Problem gives a structure theory for connected locally compact groups.

Invariant measure and the group algebra

Group algebra:
$L^1(G) = \{\phi : G \to \mathbb{C} \mid \phi \text{ measurable}, \int_G |\phi| \, \mathrm{d}m < \infty\}$.
Convolution product: $\phi * \psi(x) = \int_G \phi(xy^{-1})\psi(y) \, \mathrm{d}m(y)$.

Example:
$G = (\mathbb{T}, \times)$, where $\mathbb{T} = \{e^{ix} \mid -\pi \leq x \leq \pi\}$ is the circle.
The homomorphisms $e^{ix} \mapsto e^{inx}$, $n \in \mathbb{Z}$, are *characters* on $\mathbb{T}$.
$f \sim \sum_{n \in \mathbb{Z}} c_n e^{inx}$ is the *Fourier series* for f. (Convergence?)
Fourier transform is the analogue of Maschke's Theorem.

$\mathbb{T}$ is a compact, abelian group but ...
the Haar integral makes it possible to extend the definitions to all locally compact groups.
How far do the theorems extend to non-compact and non-abelian groups?

Groups with an invariant measure

Non-compact abelian: direct sum replaced by direct integral of 1-dimensional representations. All irreducible representations are 1-dimensional characters.

Non-abelian compact: direct sums retained, concept of characters of non-abelian groups extends. All irreducible representations are finite-dimensional.

Non-abelian non-compact: aim to decompose the regular representation on $L^2(G)$ as a direct integral, and ask whether every irreducible can be found within the regular representation.

Amenable groups

More functions may be averaged if the class of groups is restricted further.

G is *amenable* if there is linear $M : L^\infty(G) \to \mathbb{C}$ such that:
1) $M(\phi) \geq 0$ if $\phi \geq 0$;
2) $M(\mathbf{1}) = 1$; and
3) $M(\phi_x) = M(\phi)$ for every $x \in G$ and $\phi \in L^\infty(G)$.
M is called a *left-invariant mean* on G.

Equivalently, G is amenable if there is a *finitely* additive, positive, left-invariant measure $m : \mathcal{P}(G) \to \mathbb{R}^+$ with $m(G) = 1$.

Example: if G is compact, the Haar measure is a mean.

The Banach-Tarski Paradox

Does surface area on the sphere extend to a *finitely* additive, rotation invariant measure defined on *all* subsets?

The Banach-Tarski Paradox (AC) shows that the answer is *no*. It partitions the sphere into five subsets which may be rotated to cover the sphere twice.

The idea of the proof is that $SO(3)$ has a non-abelian free subgroup and that such a paradoxical decomposition may be found for the free group.

Paradoxical decompositions

The group G has a *paradoxical decomposition* if there are: a partition $\{P_1, \ldots, P_m, Q_1, \ldots, Q_n\}$ of G and elements $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ in G such that $\{x_1P_1, \ldots, x_mP_m\}$ and $\{y_1Q_1, \ldots, y_nQ_n\}$ are partitions of G.
A group having a paradoxical decomposition is strongly infinite.

Example: a non-abelian free group has a paradoxical decomposition.

Theorem (Tarski)

The group G is amenable if and only if it does not have a paradoxical decomposition.

Stability properties

Theorem

Let G be amenable. Then every subgroup and every quotient group of G is amenable.

Theorem

1. *Suppose that G has a normal subgroup N such that N and G/N are amenable. Then G is amenable.*
2. *Suppose that $G = \bigcup G_\alpha$ where G_α is amenable for each α. Then G is amenable.*

Conjecture (von Neumann)

Every non-amenable discrete group has a non-abelian free subgroup. *False: Grigorchuk, Olshanskii*

Algebras and representations

Amenability has many other characterisations, including in terms of fixed-point theorems. Here are some others.

Theorem

The following are equivalent:

1. *G is amenable;*
2. *every irreducible unitary representations of G is weakly contained in the regular representation on $L^2(G)$; and*
3. *the trivial representation is weakly contained in the regular representation.*

Theorem

G is amenable if and only if $\mathcal{H}^1(L^1(G), X^*) = (0)$ for every Banach $L^1(G)$-bimodule X.

Random walks

A probability measure is a (countably additive) mean.
The convolution product of two probability measures on G is a probability measure. The convolution powers μ^{*k}, $k = 1, 2, \ldots$, are the probability distributions for a random walk on G.

Theorem (Kawada & Ito)

Let G be a compact group and μ be a probability measure on G that is not supported on a proper subgroup of G. Then $\frac{1}{n+1}\sum_{k=0}^{n} \mu^{*k}$ converges weakly to the Haar measure on G.

Cannot hold for non-compact groups but weaker statements do hold for amenable groups.

μ-Harmonic functions

A *μ-harmonic function* ϕ on a group G is a function that satisfies $\mu * \phi = \phi$.
The probability measure μ on the group G is *Liouville* if every bounded μ-harmonic function on G is constant.
Example: compact, abelian and nilpotent groups have the property that every probability measure not supported on a proper subgroup is Liouville.

Theorem

The (σ-compact) locally compact group G is amenable if and only if it supports a Liouville probability measure.