Colloquium on
Moments of Ramanujan's generalized elliptic integrals and extensions of Catalan's constant

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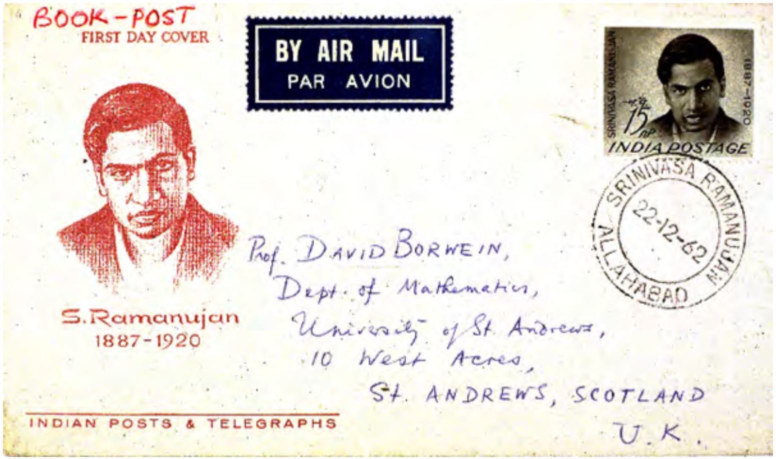
1 Abstract

- We investigate the moments of Ramanujan's alternative elliptic integrals and of related hypergeometric functions. This involved quite a lot of symbolic computation.
- Along the way we are able to give some surprising closed forms for Catalan-related constants and new hypergeometric identities.



- We also tell some of the striking history around these matters.

2 Introduction and background

As in [6, pp.178-179] for $0 \le s < 1/2$, $0 \le k \le 1$, let

$$K^s(k) \;:=\; \frac{\pi}{2}\, {}_2F_1\left(\left.\begin{matrix} \frac12 - s, \frac12 + s \\ 1 \end{matrix}\right| k^2\right) \tag{1}$$

and

$$E^s(k) \;:=\; \frac{\pi}{2}\, {}_2F_1\left(\left.\begin{matrix} -\frac12 - s, \frac12 + s \\ 1 \end{matrix}\right| k^2\right). \tag{2}$$

We use the standard notation for *hypergeometric functions*, namely

$${}_2F_1\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ is the rising factorial or *Pochhammer symbol*; so $(1)_n = n!$.

Likewise

$${}_3F_2\left(\left.\begin{matrix} a, b, c \\ d, e \end{matrix}\right| z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{z^n}{n!}.$$

We are interested in the moments given by

$$K_n = K_{n,s} := \int_0^1 k^n K^s(k)\,\mathrm{d}k, \qquad E_n = E_{n,s} := \int_0^1 k^n E^s(k)\,\mathrm{d}k. \tag{3}$$

for both integer and real values of n.

Note that $K^s = K^{(-s)}$. Euler's transform [3, (2.2.7] and a *contiguous relation* give:

$$E^{(-s)} = \frac{4s\,(1-k^2)}{2s-1}\,K^s + \frac{2s+1}{2s-1}\,E^s.$$

The corresponding integral form of K^s (due to Euler) is

$$K^s(k) = \frac{\cos(\pi s)}{2}\int_0^1 \frac{t^{s-1/2}}{(1-t)^{1/2+s}(1-k^2\,t)^{1/2-s}}\mathrm{d}t \tag{4}$$

$$= \cos(\pi\, s)\int_0^{\pi/2} \frac{\tan^{2s}(\theta)}{\left(1-k^2\,\sin^2\theta\right)^{1/2-s}}\,\mathrm{d}\theta. \tag{5}$$

- The latter has the nice feature of looking like the cleanest classical definition when $s = 0$: in which case K gives the period of a *pendulum* and E gives the arclength of an *ellipse* (hence the name) [6, Ch. 1].

- Many more forms for K^s, E^s can be obtained from http://dlmf.nist.gov/15.6.

A key early result, due to Gauss (1812), when $\mathrm{Re}(c-a-b) > 0$ is the closed form

$${}_2F_1\left(\left.{a,b \atop c}\right|1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{6}$$

- For ${}_3F_2$'s such closed forms have been intensely studied and evaluations are the exception not the rule.

There are four values for which these integrals are truly special:

$$s \in \Omega := \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}\right\}.$$

That is, when $\cos^2(\pi s)$ is rational.

- These are Ramanujan's *alternative elliptic integrals* as displayed in [10] and first decoded in [6].

  A comprehensive study is given in [4]. (See also [9] and [2].)

- These four cases are precisely those which produce modular functions [6, §5.5].

- Their study is currently experiencing a renewal of interest, especially regarding related elliptic series for $1/\pi$ [5], [6, §5.5] and [7]. For example, Ramanujan's ($s = 0$) series

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}}. \tag{7}$$

  allows the computation of the second half of a binary digit-string to be computed without the first half!

2.1 Reciprocal series for π

Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910 [5, 6]. The most famous, with $s = 1/4$ is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}. \tag{8}$$

- Each term of (8) adds **eight correct digits**. Gosper used (8) for the computation of a then-record 17 million digits of π in **1985**—thereby completing my first proof of (8) [6, Ch. 3].

Shortly thereafter, David and Gregory Chudnovsky found the following variant, which uses $s = 1/3$ and lies in the quadratic number field $Q(\sqrt{-163})$ rather than $Q(\sqrt{58})$:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!\,(13591409 + 545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \tag{9}$$

- Each term of (9) adds **14 correct digits**. The brothers used this formula several times, culminating in a **1994** calculation of π to over four billion decimal digits. Their extraordinary story was told in a prizewinning *New Yorker* article on *The Mountains of Pi* by Richard Preston.

- In late **2009**, equation (9) was used again for the then record computation of π to **2.7 trillion places**. In consequence, Fabrice Bellard has provided access to two trillion-digit integers whose ratio is bizarrely close to π.

- On August 6th, **2010** Shigeru Kondo and Alex Yee announced a new record of **5,000,000,000,000** decimal digits computed on Kondo's home-built $18,000 machine with 20 hard disks in 90 days.

  See www.numberworld.org/misc_runs/pi-5t/details.html.

- They used (9) with confirmation in 64 hours via the *BBP algorithm* for π based on the formula

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right), \tag{10}$$

  which can compute a string of digits in *hex* starting at say the five trillion digit mark

  - For the whole story come to my public lecture on *Life of Pi* in Brisbane at the AustMS meeting on September 28.

2.2 Classical results

The coupling equation between E^s and K^s is given in [6, p. 178] and can be derived from the generalized hypergeometric differential equation (see http://dlmf.nist.gov/15.10). It is

$$E^s = (1-k^2)\,K^s + \frac{k(1-k^2)}{1+2s}\frac{\mathrm{d}}{\mathrm{d}k}K^s. \tag{11}$$

Integrating this by parts leads to

$$K_{2,s} = \frac{(1+2s)\,E_{0,s} - 2s\,K_{0,s}}{2-2s}. \tag{12}$$

In the same fashion, multiplying by k^n before integrating the coupling provides a recursion for $K_{n+2,s}$:

$$K_{n+2,s} = \frac{(n-2\,s)\,K_{n,s} + (1+2\,s)\,E_{n,s}}{n+2\,(1-s)}. \tag{13}$$

We also consider the *complementary* integrals:

$$K'^s(k) := K^s(\sqrt{1-k^2}) \quad \text{and} \quad E'^s(k) := E^s(\sqrt{1-k^2}).$$

The four integrals then satisfy a version of *Legendre's identity*

$$E^s K'^s + K^s E'^s - K^s K'^s = \frac{\pi}{2}\frac{\cos(\pi s)}{1+2s} \tag{14}$$

for all $0 \le k \le 1$.

In [6, pp. 188-89] the moments are determined for the classical case of $s = 0$ which give the original complete elliptic integrals K, E. These are linked by the equations (see [6, p. 9])

$$E = (1 - k^2)\, K + k(1 - k^2)\frac{\mathrm{d}K}{\mathrm{d}k}, \tag{15}$$

which is (11) with $s = 0$ and

$$E = K + k\frac{\mathrm{d}E}{\mathrm{d}k}, \tag{16}$$

from which we derive the following recursions:

Theorem 1 (s=0) *For $n = 0, 1, 2, \ldots$*

$$(a)\ K_{n+2} = \frac{nK_n + E_n}{n+2} \quad \text{and} \quad (b)\ E_n = \frac{K_n + 1}{n+2}. \tag{17}$$

The recursion holds for real n.

Moreover,

$$K_0 = 2\,G, \qquad K_1 = 1, \tag{18}$$

$$E_0 = G + \frac{1}{2}, \qquad E_1 = \frac{2}{3}. \tag{19}$$

Here

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = L_{-4}(2)$$

is the *Catalan* constant whose irrationality is still not proven.

- This ignorance is part of my motivation for the study.

  The current record for computation is **31.026 billion digits** in **2009**.

  Computations often use the following central binomial formula due to Ramanujan [6, last formula] or its recent generalizations:

$$\frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} + \frac{\pi}{8} \log(2+\sqrt{3}) = G. \tag{20}$$

- `PSLQ` shows that if G is rational it has an `enormous` denominator.

- Amusingly in [1] Adamchik uses the moment evaluation above as a definition of G !

- We shall explore various ways to obtain the initial values.

- One may also profitably study fractional moments, see below and [1].

3 Basic results

We commence with various fundamental representations.

3.1 Hypergeometric closed forms

A concise closed form for the moments is

Theorem 2 (Hypergeometric forms) *For* $0 \le s < \frac{1}{2}$ *we have*

$$K_{n,s} = \frac{\pi}{2(n+1)}\, {}_3F_2\left({\frac{1}{2}-s, \frac{1}{2}+s, \frac{n+1}{2} \atop 1, \frac{n+3}{2}} \middle| 1 \right), \tag{21}$$

$$E_{n,s} = \frac{\pi}{2(n+1)}\, {}_3F_2\left({-\frac{1}{2}-s, \frac{1}{2}+s, \frac{n+1}{2} \atop 1, \frac{n+3}{2}} \middle| 1 \right). \tag{22}$$

These hold in the limit for $s = \frac{1}{2}$.

Proof. To establish (21) and (22), we begin with

$$\begin{aligned} \int_0^1 x^{u-1}(1-x)^{v-1}\, {}_2F_1\left({a, 1-a \atop b} \middle| x \right) \mathrm{d}x &= \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(b)_n n!} \int_0^1 x^{n+u-1}(1-x)^{v-1}\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n(u)_n}{(b)_n(u+v)_n n!} \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \\ &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\, {}_3F_2\left({a, 1-a, u \atop b, u+v} \middle| 1 \right). \end{aligned} \tag{23}$$

Similarly,

$$\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\,{}_3F_2\left(\left.\begin{matrix}a, -a, u\\ b, u+v\end{matrix}\right|1\right) = \int_0^1 x^{u-1}(1-x)^{v-1}\;{}_2F_1\left(\left.\begin{matrix}a, -a\\ b\end{matrix}\right|x\right)\,\mathrm{d}x.$$

By applying these to (1) and (2) we immediately get (21) and (22). □

- As long as $0 < s < 1/2$, the first series (21) is *Saalschützian* [11]. That is, the denominator parameters add to one more than those in the numerator, but is not *well poised*, and can be reduced to Gamma functions only for $n = \pm 1$ (with $n = -1$ a pole) since then it reduces to a ${}_2F_1$.

- The second (22) is not even Saalschützian, although it is *nearly well poised* (whose definition [11] we do not need) and also can be reduced to Gamma functions for $n = \pm 1$.

Thus, for $|s| < 1/2$ we find

$$K_{1,s} = \frac{\cos(\pi s)}{1-4s^2}, \qquad E_{1,s} = \frac{2}{2s+3}\,\frac{\cos(\pi s)}{1-4s^2}. \tag{24}$$

In general we obtain:

Theorem 3 (Odd moments) *For odd integers $2m+1$ and $m = 0,1,2,\ldots,$*

$$K_{2m+1,s} = \frac{\cos(\pi s)\, m!^2}{4\,\Gamma\left(\frac{3}{2}-s+m\right)\Gamma\left(\frac{3}{2}+s+m\right)} \sum_{k=0}^{m} \frac{\Gamma\left(\frac{1}{2}-s+k\right)\Gamma\left(\frac{1}{2}+s+k\right)}{k!^2}. \tag{25}$$

Proof. In terms of the *Legendre function*,

$${}_2F_1\left(\left.\begin{matrix} a, 1-a \\ 1 \end{matrix}\right| z\right) =: P_{-a}(1-2z),$$

where $y = P_\nu(x) = {}_2F_1\left(\left.\begin{smallmatrix} -\nu,\nu+1 \\ 1 \end{smallmatrix}\right| \frac{1-x}{2}\right)$ solves $(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \nu(\nu+1)y = 0$. In consequence we may deduce that

$$\begin{aligned} {}_2F_1\left(\left.\begin{matrix} a, 1-a \\ 1 \end{matrix}\right| z\right) \quad &= \quad \frac{\sin \pi a}{\pi} \sum_{k=0}^{\infty} \frac{(a)_k(1-a)_k}{k!^2}\,(1-z)^k \\ &\times \quad \left\{2\Psi(1+k) - \Psi(a+k) - \Psi(1-a+k) - \log(1-z)\right\}, \end{aligned}$$

where

$$\Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) \mathrm{d}t.$$

- Here γ is Euler's constant and Ψ the *digamma* function (one of the workhorses of the special function world). Also

$$\Psi(n+1) = H_n - \gamma \quad \text{where} \quad H_n := 1 + 1/2 + \cdots + 1/n.$$

Now, by integrating the series term-by-term and noting integral (23), we have

$$\begin{aligned}\int_0^1 z^{n-1}{}_2F_1\left(\left.\begin{matrix}a, 1-a\\1\end{matrix}\right|z\right)\,\mathrm{d}z &= \frac{1}{n}\,{}_3F_2\left(\left.\begin{matrix}a, 1-a, n\\1, n+1\end{matrix}\right|1\right)\\ &= \frac{(n-1)!\sin(\pi a)}{\pi}\sum_{k=0}^{\infty}\frac{(a)_k(1-a)_k}{k!(k+n)!}\\ &\times\ \{\Psi(1+k)+\Psi(n+1+k)-\Psi(a+k)-\Psi(1-a+k)\}\,.\end{aligned}$$

We note in passing that this offers an apparently new approach for summing this class of hypergeometric series; we exploit (23) again in section 6.4.

Thus, for example, by *creative telescoping*, one finds for any positive integer n that

$${}_3F_2\left(\left.\begin{matrix}a, 1-a, n\\1, n+1\end{matrix}\right|1\right) = \frac{\Gamma(n)\,\Gamma(1+n)}{\Gamma(a+n)\,\Gamma(1-a+n)}\sum_{k=0}^{n-1}\frac{(a)_k(1-a)_k}{k!^2}. \tag{26}$$

Now, with $n = m+1$ in (26) we conclude the proof of Theorem 3. □

Similarly,

$$\begin{aligned}{}_2F_1\left(\left.\begin{matrix}a, -a\\1\end{matrix}\right|z\right) &= \frac{\sin(\pi a)}{\pi a}\Big\{1 - a^2\sum_{k=0}^{\infty}\frac{(a+1)_k(1-a)_k}{k!(k+1)!}(1-z)^{k+1}\\ &\times\ [\Psi(a+1+k)+\Psi(1-a+k)-\Psi(k+1)-\Psi(k+2)+\ln(1-z)]\Big\}.\end{aligned}$$

For $m = 0$, Theorem 3 reduces to the evaluation given in (24). In general, it gives $\cos(\pi s)$ times a rational function. An equivalent, rather pretty, partial fraction decomposition is

$$K_{2m+1,s} = \frac{\cos \pi s}{2} \sum_{k=0}^{m} \frac{m!^2}{(m-k)!(m+k+1)!} \left(\frac{1}{2k+1-2s} + \frac{1}{2k+1+2s} \right). \tag{27}$$

This can easily be confirmed inductively, using say (71).

- For $s = 0$ this result originates with Ramanujan.
- Victor Adamchik [1] reprises its substantial history and extensions which include a formula due independently to Bailey and Hodgkinson in 1931 which subsumes (26).

It is

$${}_3F_2\left(\left. \begin{matrix} a, b, c+1 \\ a+b+n \end{matrix} \right| 1 \right) = \frac{\Gamma(n)\Gamma(a+b+n)}{\Gamma(a+n)\Gamma(b+n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k}{(c)_k (1)_k}. \tag{28}$$

- The elliptic case of $a = b = 1/2, c = 1$ was in Ramanujan's first letter to Hardy.

Example 1 (Digamma consequences) For $0 < a < 1/2$, consequences are neatly given using: $\gamma(\nu) := \frac{1}{2}\left\{\Psi\left(\frac{\nu+1}{2}\right) - \Psi\left(\frac{\nu}{2}\right)\right\}$. Moreover

$$\begin{aligned}\gamma\left(\frac{1}{2}\right) &= \frac{\pi}{2}, \qquad \gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sqrt{2}} - \sqrt{2}\log(\sqrt{2}-1),\\ \gamma\left(\frac{1}{3}\right) &= \frac{\pi}{\sqrt{3}} + \log 2, \qquad \gamma\left(\frac{1}{6}\right) = \pi + \sqrt{3}\log(2+\sqrt{3}).\end{aligned}$$

$$\sum_{k=0}^{\infty} \frac{(a)_k(1-a)_k}{\left(\frac{3}{2}\right)_k k!}\left[\Psi(k+1) + \Psi\left(k+\frac{3}{2}\right) - \Psi(k+a) - \Psi(k+1-a)\right] = \frac{2\gamma(a) - \pi\csc(\pi a)}{1-2a}.$$

This in turn gives

$$ {}_3F_2\left(\begin{matrix} a, 1-a, \frac{1}{2} \\ 1, \frac{3}{2}\end{matrix}\middle| 1\right) = \frac{2\sin(\pi a)}{\pi(1-2a)}\gamma(a) - \frac{1}{1-2a}. \tag{29}$$

Taking the limit as $a \to 1/2$ in (29) gives two useful specializations:

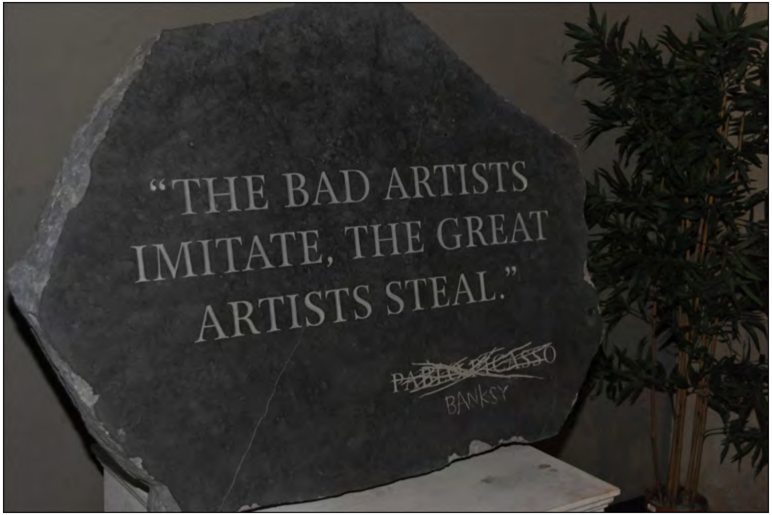
$$(a) \qquad {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2}\end{matrix}\middle| 1\right) = \frac{4G}{\pi} \tag{30}$$

$$(b) \qquad \Psi'\left(\frac{1}{4}\right) = \pi^2 + 8\,G, \tag{31}$$

with (a) being known and very useful but far from obvious. ◇

The corresponding form for $E_{2m+1,s}$ is similar but less satisfactory:

$$E_{2m+1,s} = \frac{\pi}{4\,(m+1)} \frac{1}{\Gamma(\frac{3}{2}+s)\Gamma(\frac{1}{2}-s)} + \frac{\pi}{4} \frac{m!}{\Gamma(\frac{1}{2}+s)\Gamma(-\frac{1}{2}-s)} \times \sum_{k=0}^{\infty} \frac{(\frac{3}{2}+s)_k(\frac{1}{2}-s)_k}{k!(k+m+2)!} \left\{ \Psi\left(\frac{3}{2}+s+k\right) + \Psi\left(\frac{1}{2}-s+k\right) - \Psi\,(k+1) - \Psi\,(3+m+k) \right\}. \tag{32}$$



Homage to Lobachevsky

- Can one do better?

Example 2 (Other special values) For each $s \neq 0$ there are also two special values of r for which $K_{r,s}$ also reduce to a ${}_2F_1$. They are obtained by solving $r + 3/2 = 1/2 \pm s$.

This and similar calculations for $E_{n,s}$ yield

$$K_{(-2\pm 2s),s} = -\frac{\cos(\pi s)}{(1 \mp 2s)^2}, \tag{33}$$

$$E_{(-2-2s),s} = -\frac{2}{(1+2\,s)}\frac{\cos(\pi\,s)}{(1-2\,s)^2}, \tag{34}$$

$$E_{(-4-2s),s} = -\frac{2}{(1+2\,s)}\frac{\cos(\pi\,s)}{(3+2\,s)^2}. \tag{35}$$

The r-recursions given above in (13) for $K_{r,s}$ and below in equation (73) for $E_{r,s}$ extend this to values of $r + 2n$, for n integral. ◇

Example 3 (Alternative moment expansions) We also obtain

$$
\begin{aligned}
K_{0,s} &= \frac{\cos(\pi s)}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}+s\right)_n \left(\frac{1}{2}-s\right)_n}{n! \left(\frac{3}{2}\right)_n} \left\{ \Psi(n+1) + \Psi\left(\frac{3}{2}+n\right) - \Psi\left(\frac{1}{2}+n+s\right) - \Psi\left(\frac{1}{2}+n-s\right) \right\}, \\
E_{0,s} &= \frac{\cos(\pi s)}{2s+1} + \cos(\pi s) \frac{2s+1}{6} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}+s\right)_n \left(\frac{1}{2}-s\right)_n}{n! \left(\frac{5}{2}\right)_n} \\
&\times \left\{ \Psi(n+1) + \Psi\left(\frac{5}{2}+n\right) - \Psi\left(\frac{3}{2}+n+s\right) - \Psi\left(\frac{1}{2}+n-s\right) \right\}.
\end{aligned}
$$

$\diamond$

3.1.1 Half-integer values of s

For $s = m + 1/2$, and $m, n = 0, 1, 2 \ldots$ we can obtaining a terminating representation

$$\begin{aligned} K_{n,m+1/2} &= \frac{\pi}{2(n+1)} \, {}_3F_2\left(\left. \begin{matrix} -m, m+1, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix} \right| 1 \right) \\ &= \frac{(-1)^m \pi}{4} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2} - m\right) \Gamma\left(\frac{n+3}{2} + m\right)}, \end{aligned} \tag{36}$$

and likewise

$$E_{n,m+1/2} = \frac{\pi}{2} \sum_{k=0}^{m+1} \frac{(-m-1)_k \, (m+1)_k}{(n+1+2k) \, k!^2}. \tag{37}$$

3.2 The complementary integrals

By contrast the complementary integral moments are less recondite.

Theorem 4 (Complementary moments) *For* $n = 0, 1, 2, \ldots$ *and* $0 \le s < \frac{1}{2}$ *we have*

$$K'_{n,s} = \frac{\pi}{4} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2-2s}{2}\right)\Gamma\left(\frac{n+2+2s}{2}\right)} \tag{38}$$

$$E'_{n,s} = \frac{\pi}{2(n+1)} \frac{\Gamma^2\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2-2s}{2}\right)\Gamma\left(\frac{n+4+2s}{2}\right)}. \tag{39}$$

These hold in the limit for $s = \frac{1}{2}$. *In particular, recursively we obtain for all* real n*:*

$$\text{(a)} \quad K'_{n+2,s} = \frac{(n+1)^2}{(n+2)^2 - 4s^2} K'_{n,s}, \qquad \text{(b)} \quad E'_{n,s} = \frac{n+1}{n+2+2\,s} K'_{n,s}, \tag{40}$$

where

$$\text{(c)} \quad K'_{0,s} = \frac{\pi}{4} \frac{\sin(\pi\, s)}{s}, \qquad \text{(d)} \quad K'_{1,s} = \frac{\cos(\pi s)}{1 - 4s^2}. \tag{41}$$

Proof. To establish (38) we recall that

$$K'_s = \frac{\pi}{2}\, {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s\\ 1\end{matrix}\right| 1-k^2\right), \tag{42}$$

and so

$$\begin{aligned}
K'_{n,s} &= \frac{\pi}{2}\int_0^1 x^n\, {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s\\ 1\end{matrix}\right| 1-x^2\right) \mathrm{d}x \\
&= \frac{\pi}{4}\int_0^1 x^{\frac{n+1}{2}-1}{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s\\ 1\end{matrix}\right| 1-x\right) \mathrm{d}x \\
&= \frac{\pi}{4}\int_0^1 (1-x)^{\frac{n+1}{2}-1}\, {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s\\ 1\end{matrix}\right| x\right) \mathrm{d}x \\
&= \frac{\pi}{2(n+1)}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s, 1\\ 1, \frac{n+3}{2}\end{matrix}\right| 1\right) \\
&= \frac{\pi}{2(n+1)}\, {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s\\ \frac{n+3}{2}\end{matrix}\right| 1\right),
\end{aligned}$$

which is summable, by Gauss' formula (6), to the desired result.

The proof of (39) is similar; and the recursions follow. □

Example 4 (Complementary closed forms) Thence, with $s = 0$ and $n = 0, 1$ we recover

$$K_0' = \frac{\pi^2}{4}, \quad E_0' = \frac{\pi^2}{8}, \quad K_1' = 1, \quad E_1' = \frac{2}{3},$$

as discussed in [6, p.98]. Correspondingly

$$K'_{0,1/6} = \frac{3\pi}{4}, \quad K'_{1,1/6} = \frac{9\sqrt{3}}{16}, \quad E'_{0,1/6} = \frac{9\pi}{28}, \quad K'_{1,1/6} = \frac{27\sqrt{3}}{80},$$

$$K'_{0,1/3} = \frac{3\sqrt{3}\pi}{8}, \quad K'_{1,1/3} = \frac{9}{10}, \quad E'_{0,1/3} = \frac{9\sqrt{3}\pi}{64}, \quad E'_{1,1/3} = \frac{27}{55}.$$

We note that π, not π^2 appears in these evaluations, since in (41)(c),

$$\frac{\sin(\pi\, s)}{s} \to \pi$$

as $s \to 0$. $\diamond$

3.2.1 Connecting moments and complementary moments

We first remark that a comparison of Theorems 3 and 4 shows that for all s we have

$$K'_{1,s} = K_{1,s} \qquad \text{and} \qquad E'_{1,s} = E_{1,s}.$$

The formula

$$\int_0^1 K(k)\,\frac{\mathrm{d}k}{1+k} = \int_0^1 K\left(\frac{1-h}{1+h}\right)\frac{\mathrm{d}h}{1+h} = \frac{1}{2}\int_0^1 K'(k)\,\mathrm{d}k, \tag{43}$$

is recorded in [6, p. 199]. It is proven by using the *quadratic transformation* [6, Thm 1.2 (b), p. 12] for the second equality and a substitution for the first. This implies

$$2\sum_{n=0}^{\infty}(-1)^n K_n = \frac{\pi^2}{4} = K'_0, \tag{44}$$

on appealing to Theorem 4.

The corresponding identity for $s = 1/6$ is best written

$$\int_0^1 {}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\right| 1-t^3\right)\mathrm{d}t = 3\int_0^1 {}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\right| t^3\right)\frac{\mathrm{d}t}{1+2\,t}, \tag{45}$$

which follows analogously from the *cubic transformation* [8, Eqn 2.1] and a change of variables. This is a beautiful counterpart to (43) especially when the latter is written in hypergeometric form:

$$\int_0^1 {}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\right| 1-k^2\right)\mathrm{d}k = 2\int_0^1 {}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\right| k^2\right)\frac{\mathrm{d}k}{1+k}, \tag{46}$$

- We further evaluate equation (45) in (93) of section 6.4.

Additionally, [6, p. 188] outlines how to derive

$$\int_0^1 \frac{K(k)}{\sqrt{1-k^2}}\,\mathrm{d}k = K\left(\frac{1}{\sqrt{2}}\right)^2.$$

Using the same technique, we generalize this to

$$\int_0^1 \frac{K^s(k)}{\sqrt{1-k^2}}\,\mathrm{d}k = K^s\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\cos^2(\pi s)}{16\pi}\Gamma^2\left(\frac{1}{4}+\frac{s}{2}\right)\Gamma^2\left(\frac{1}{4}-\frac{s}{2}\right). \tag{47}$$

Here we have used Gauss' ${}_2F_1$ summation theorem (6) for the evaluation

$$K^s\left(\frac{1}{\sqrt{2}}\right) = \frac{\cos \pi s}{4}\,\beta\left(\frac{1}{4}+\frac{s}{2}, \frac{1}{4}-\frac{s}{2}\right).$$

By the generalized Legendre identity (14), which simplifies as the complementary integrals coincide with the original ones at $1/\sqrt{2}$, we obtain

$$E^s\left(\frac{1}{\sqrt{2}}\right) = \frac{K^s\left(\frac{1}{\sqrt{2}}\right)}{2} + \frac{\pi\cos(\pi s)}{4(2s+1)K^s(\frac{1}{\sqrt{2}})}.$$

3.3 Analytic continuation of results

We finish this section by recalling a useful—almost illegally so—theorem:

Theorem 5 (Carlson (1914)) *Let f be analytic in the right half-plane $\Re z \geq 0$ and of* exponential type *(meaning that $|f(z)| \leq Me^{c|z|}$ for some M and c), with the additional requirement that*

$$|f(z)| \leq Me^{d|z|}$$

for some $d < \pi$ on the imaginary axis $\Re z = 0$. If $f(k) = 0$ for $k = 0, 1, 2, \ldots$ then $f(z) = 0$ identically.

- $\sin(z)$ show that the growth condition is optimal.
- Carlson's theorem [12, 5.81] allow us to prove that many of the results in this paper hold *generally* as soon as they hold for integer n.
- For example, equations (70) or (71) hold generally as soon as the integral cases hold: once we check growth on the imaginary axis which is easy for hypergeometric functions.
- This matter is discussed at some length in [3, Thm 2.8.1 and sequel] in which an elegant 1941 proof by Selberg is given for the case in which $f(x)$ is bounded for $\mathrm{Re}(z) \geq 0$.

4 Closed form initial-values for various s

Many results work for all s but (as we have seen) a few others are more satisfactory when $s \in \Omega$ – since these four K^s are the only modular functions ([6, Prop 5.7], [8]) amongst the generalized elliptic integrals K^s.

Empirically, we discovered an algebraic relation

$$2(1+s)\,E_{0,s} - (1+2s)\,K_{0,s} = \frac{\cos(\pi s)}{1+2s}. \tag{48}$$

Equivalently, we exhibit a parametric reciprocal series for π:

$$\frac{2\cos\pi s}{(1+2s)\pi} = (2+2s)\,{}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}+s, -\frac{1}{2}-s \atop 1, \frac{3}{2}}\right|1\right) - (1+2s)\,{}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}+s, \frac{1}{2}-s \atop 1, \frac{3}{2}}\right|1\right). \tag{49}$$

On using (12) to eliminate $E_{0,s}$ in (48) this becomes

$$K_{2,s} = \frac{K_{0,s} + \cos(\pi\, s)}{4-4s^2} \tag{50}$$

which in turn is a special case of (71) with $r = \frac{1}{2}$—as is justified by Carlson's Theorem 5—thus proving our observation.

- Hence, to resolve all integral values, for a given s we are left looking for satisfactory representations only for $K_{0,s}$.

We write

$$G_s := \frac{1}{2} K_{0,s} = \frac{\pi}{4}\, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}-s, \frac{1}{2}+s \\ 1, \frac{3}{2} \end{matrix} \right| 1 \right).$$

and call this the associated or *generalized Catalan* constant.

- What is a generalization for G_s of the central binomial formula (20) for G?

- For various reasons, the results for $s = 1/6$ are especially interesting. This is the case corresponding to the cubic AGM [8].

From (21) we obtain

$$\begin{aligned} K_{0,s} = \frac{\pi}{2}\, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}-s, \frac{1}{2}+s \\ 1, \frac{3}{2} \end{matrix} \right| 1 \right) &= \frac{\cos(\pi s)}{2} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n+s\right)\Gamma\left(\frac{1}{2}+n-s\right)}{(2n+1)\,(n!)^2} \\ &= \frac{\cos(\pi s)}{2} \sum_{n=0}^{\infty} \beta\left(n+\frac{1}{2}-s, n+\frac{1}{2}+s\right) \frac{\binom{2n}{n}}{2n+1} \\ &= \frac{\cos(\pi s)}{4} \int_0^1 \frac{\arcsin\left(2\sqrt{t-t^2}\right)}{t^{1+s}\,(1-t)^{1-s}} \mathrm{d}t \\ &= \frac{\cos(\pi s)}{2} \int_0^{\pi/2} \left\{ \tan^{2s}\left(\frac{\theta}{2}\right) + \cot^{2s}\left(\frac{\theta}{2}\right) \right\} \frac{\theta}{\sin\theta}\, \mathrm{d}\theta. \end{aligned} \tag{51}$$

1. This uses the definition directly, see also [6, Prop 5.6], to attain the first identity after writing the rising factorials in terms of the β-function:

$$\beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t.$$

2. We exchange integral and sum to arrive at the penultimate integral. Moving the integral to $[-1/2, 1/2]$ and then making various trig substitutions, we arrive at the final result in (51).

For example, we have

$$K_{0,0} = \int_0^{\pi/2} \frac{\theta}{\sin\theta}\,\mathrm{d}\theta = 2G.$$

The final equality has various derivations [6, 1]. These include contour integration as explored in section 5.

- If we now make the trigonometric substitution $t = \tan(\theta/2)$ in (51), and integrate the two resulting terms separately, we arrive at a central result.

Theorem 6 (Generalized Catalan values for $0 \leq s \leq \frac{1}{2}$)

$$\begin{aligned} K_{0,s} &= \cos(\pi s) \int_0^1 \left(t^{2s-1} + t^{-2s-1}\right) \arctan t \, \mathrm{d}t \\ &= \frac{\cos(\pi s)}{8s} \left\{ \Psi\left(\frac{3-2s}{4}\right) + \Psi\left(\frac{1+2s}{4}\right) - \Psi\left(\frac{1-2s}{4}\right) - \Psi\left(\frac{3+2s}{4}\right) \right\} \quad (52) \\ &= \frac{\cos \pi s}{4\, s} \left\{ \Psi\left(\frac{s}{2} + \frac{1}{4}\right) - \Psi\left(\frac{s}{2} + \frac{3}{4}\right) \right\} + \frac{\pi}{4\, s} = 2\, G_s. \quad (53) \end{aligned}$$

- For $s = 0$, L'Hôpital's rule provides

$$\frac{1}{2}\, K_{0,0} = \frac{1}{16} \Psi'\left(\frac{1}{4}\right) - \frac{1}{8} \Psi'\left(\frac{3}{4}\right)$$

  which is precisely G.

- The digamma expression in (52) simplifies entirely when $s \in \Omega$ to the forms originally discovered in the next section. We finally obtain complete evaluations for $s \in \Omega$ as was our goal.

Corollary 1 (Generalized Catalan values for s in Ω)

$$\boxed{G_0 = G, \quad G_{1/6} = \frac{3}{4}\sqrt{3}\,\log 2, \quad G_{1/4} = \log\left(1+\sqrt{2}\right), \quad G_{1/3} = \frac{3}{8}\sqrt{3}\,\log\left(2+\sqrt{3}\right).} \quad (54)$$

- *Mathematica*, which currently knows more about the Psi function than *Maple* does, can evaluate the integral in Theorem 6 symbolically for some s.

  For example, if $s = 1/12$, after simplification we have the very nice expression:

$$G_{1/12} = 3\left(\sqrt{3}+1\right)\left\{\log\left(\sqrt{2}-1\right) + \frac{\sqrt{3}}{2}\log\left(\sqrt{3}+\sqrt{2}\right)\right\}.$$

- More generally, the evaluation requires only knowledge of $\sin(\pi s/2)$, and hence we can determine which s give a reduction to radicals.

  As a last example:

$$G_{1/5} = \frac{5}{8}\sqrt{5+2\sqrt{5}}\left\{\frac{\sqrt{5}-1}{2}\,\mathrm{arcsinh}\left(\sqrt{5+2\sqrt{5}}\right) - \mathrm{arcsinh}\left(\sqrt{5-2\sqrt{5}}\right)\right\}.$$

5 Contour integrals for $K_{0,s}$

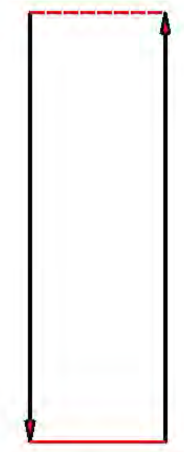
By contour integration on the infinite rectangle above $[0, \pi/2]$ (see the Figure) we obtain

$$G_0 = \frac{1}{2}\int_0^\infty \frac{t}{\cosh t}\,\mathrm{d}t = \int_0^\infty \frac{te^{-t}}{1+e^{-2t}}\,\mathrm{d}t = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2} = G. \tag{55}$$

- Here we used the geometric series, and integrated termwise the Γ-function terms we got.
- The final evaluation is definitional.
- *Maple* will *evaluate*

$$\mathbf{x} \mapsto \int_0^{\mathbf{x}} \frac{t}{\cosh t}\,\mathrm{d}t,$$

in *dilogarithmic* terms. Care to simplify it?



The contour we use

- Contour integration over the rectangle (Cauchy) provides an integral for G_s with $0 \le s < 1/2$. Namely, we integrate:

  1. From 0 to $\pi/2$.
  2. From $\pi/2$ to $\pi/2 + N\,i\infty$ for large N.
  3. From $\pi/2 + N\,i\infty$ to $0 + N\,i\infty$ and home.

We check the integral on the top of the rectangle goes to zero as $N \to \infty$ and deduce:

Theorem 7 (Contour integral for G_s) *For $0 \le s < 1/2$ we have*

$$\begin{aligned} 2\,G_s = K_{0,s} &= 2^{2s}\sin\left(2\pi s\right)\int_0^\infty \frac{(\cosh t)^{4s} - (\sinh t)^{4s}}{(\sinh 2t)^{2s+1}}\,t\,\mathrm{d}t \\ &+ \cos\left(\pi s\right)\int_0^\infty \frac{\cos\left(2\,s\arctan\left(\sinh t\right)\right)}{\cosh t}\,t\,\mathrm{d}t. \end{aligned} \tag{56}$$

- When $s = 0$ this is the previous result.
- When $s = 1/2$ the result fails $(\pi/2 = 0)$.
- When $s = 1/4$ it is especially simple ...

Example 5 (Experimentally obtained evaluations) For $s = 1/4$, equation (56) becomes

$$K_{0,1/4} = \sqrt{2}\int_0^\infty \frac{\cosh t - \sinh t}{(\sinh 2\,t)^{3/2}}\, t\mathrm{d}t + 2\,\sqrt{2}\int_0^\infty \frac{\cosh t}{(\cosh 2\,t)^{3/2}}\, t\mathrm{d}t, \tag{57}$$

with numerical value ≈ 1.7627471740392. Here for the first time the specific form of the root of unity has played a role.

Quite remarkably, when we—much as before—converted the integrand to exponential form and apply the binomial theorem we obtained Γ-function values which became:

$$\begin{aligned} G_{1/4} &= \sum_{n=0}^{\infty} \binom{-\frac{3}{2}}{n} \frac{12\,n + 8\,n^2 + 5 + (-1)^n\,(2\,n+1)^2}{8\,(n+1)^2\,(2\,n+1)^2} \\ &= \log\left(1+\sqrt{2}\right). \end{aligned} \tag{58}$$

Having thus *proven* this, we then discovered using the integer relation algorithm `PSLQ` and the *Maple* `identify` function that:

$$K_{0,1/6} = \frac{3}{2}\sqrt{3}\,\log 2, \tag{59}$$

with numerical value ≈ 1.8008492007794, and a similar evaluation:

$$K_{0,1/3} = \frac{3}{2}\sqrt{3}\,\log\left(1+\sqrt{3}\right) - \frac{3}{4}\sqrt{3}\,\log(2). \tag{60}$$

with numerical value ≈ 1.7107784916770. ◇

Example 6 (Further integrals) We have discovered additionally, using *inverse symbolic computational methods* (http://carma.newcastle.edu.au/isc2), that

$$\int_0^\infty \frac{(\cosh t)^{4/3} - (\sinh t)^{4/3}}{(\sinh t \cosh t)^{5/3}}\, t\mathrm{d}t = \frac{9}{4}\log(3),$$

and

$$\int_0^\infty \frac{(\cosh t)^{2/3} - (\sinh t)^{2/3}}{(\sinh t \cosh t)^{4/3}}\, t\mathrm{d}t = \frac{3}{2}\log\left(\frac{27}{16}\right).$$

◇

"It's a Broadway musical adapted from a YouTube video based on a Twitter tweet inspired by a Facebook comment about a text message."

- In light of Corollary 1 the discoveries of these two examples are now proven.

5.1 Contour integral based series for $K_{0,s}$

Let us write

$$K_{0,s} = \sin(2\pi s)\, S(s) + \cos(\pi s)\, C(s) \tag{61}$$

where

$$S(s) \;:=\; 2^{2s} \int_0^\infty \frac{(\cosh t)^{4s} - (\sinh t)^{4s}}{(\sinh 2t)^{2s+1}}\, t\, \mathrm{d}t \tag{62}$$

$$C(s) \;:=\; \int_0^\infty \frac{\cos(2\, s \arctan(\sinh t))}{\cosh t}\, t\, \mathrm{d}t. \tag{63}$$

- To evaluate $S(s)$ we make a substitution $u = \tanh(t)$. We obtain

$$\begin{aligned} S(s) &= \frac{1}{2} \int_0^1 (u^{-2s-1} - u^{2s-1}) \,\mathrm{arctanh}(u) \mathrm{d}u \\ &= \frac{-1}{8s} \left(2\gamma + 4\log(2) + \Psi\left(\frac{1}{2} - s\right) + \Psi\left(\frac{1}{2} + s\right) \right). \end{aligned} \tag{64}$$

- Here as above

$$\gamma := \lim_{n\to\infty} H_n - \log n$$

is the *Euler-Mascheroni* constant whose irrationality is both certain but not proven.

• To evaluate $C(s)$ we note that

$$\cos\left(2\,s\arctan\left(\sinh t\right)\right) \;=\; \cos\left(2\,s\arcsin\left(\tanh t\right)\right) = {}_2F_1\left(\left.\begin{matrix} s, -s \\ \frac{1}{2} \end{matrix}\right| \tanh^2 t\right) \tag{65}$$

and so we obtain a converging (finite if $s = 0$) series

$$C(s) \;=\; \int_0^\infty \frac{\cos\left(2\,s\arctan\left(\sinh t\right)\right)}{\cosh t}\,t\,\mathrm{d}t = \sum_{n=0}^\infty \frac{(s)_n\,(-s)_n}{\left(\frac{1}{2}\right)_n}\,\frac{\tau_n}{n!}$$

where

$$\tau_n := \int_0^\infty \frac{x^{2n}}{\left(1+x^2\right)^{n+1}}\,\mathrm{arcsinh}(x)\,\mathrm{d}x, \tag{66}$$

and where we have expanded termwise.

Moreover,

$$\tau_{m+2} = \frac{\left(13 + 8\,m^2 + 20\,m\right)\tau_{m+1} - 2\,\left(m+1\right)\left(2\,m+1\right)\tau_m}{2\,\left(m+2\right)\left(2\,m+3\right)} \tag{67}$$

where $\tau_0 = K_0 = 2G$ and $\tau_1 = E_0 = G + \frac{1}{2}$.

In particular $C(0) = 2G$.

A closed form for τ_n is easily obtained. It is

$$\tau_n = \beta\left(n+\frac{1}{2},\frac{1}{2}\right)\left\{\frac{2\,G}{\pi}+\frac{1}{4}\sum_{k=1}^{n}\frac{\Gamma\left(k\right)^2}{\Gamma\left(k+\frac{1}{2}\right)^2}\right\}. \tag{68}$$

Collecting up evaluations, we deduce that

$$\begin{aligned} K_{0,s} &= \sin\left(2\,\pi\,s\right)\left\{\frac{-1}{8s}\left(2\gamma+4\log(2)+\Psi\left(\frac{1}{2}-s\right)+\Psi\left(\frac{1}{2}+s\right)\right)\right\} \\ &+ \frac{\sin(2\pi\,s)}{\pi s}\left\{G-\pi\sum_{k=0}^{\infty}\frac{\Gamma\left(k+s+1\right)\Gamma\left(k-s+1\right)-k!^2}{8\,\Gamma\left(k+\frac{3}{2}\right)^2}\right\}, \end{aligned}$$

since on interchanging order of summation

$$\frac{\pi}{4}\cos\left(\pi\,s\right)\sum_{n=1}^{\infty}\frac{(s)_n\,(-s)_n}{n!^2}\sum_{k=1}^{n}\frac{\Gamma\left(k\right)^2}{\Gamma\left(k+\frac{1}{2}\right)^2} = -\frac{\sin 2\pi\,s}{8s}\sum_{k=1}^{\infty}\frac{\Gamma\left(k+s\right)\Gamma\left(k-s\right)-\Gamma(k)^2}{\Gamma\left(k+\frac{1}{2}\right)^2}.$$

This ultimately yields:

Theorem 8 (Contour series for G_s)

$$G_s = \frac{\sin(2\pi s)}{16s}\left(\sum_{k=1}^{\infty}\frac{\Gamma(k)^2-\Gamma(k+s)\Gamma(k-s)}{\Gamma\left(k+\frac{1}{2}\right)^2}+2\Psi\left(\frac{1}{2}\right)-2\Psi\left(s+\frac{1}{2}\right)+\pi\tan(\pi s)+\frac{8G}{\pi}\right). \tag{69}$$

Example 7 (A related series) Note for $s = 0$ we obtain precisely $G_0 = G$ as all other terms are zero. Comparing, (69) to (52) leads to a closed form for the infinite series $Q(s)$ given by

$$\begin{aligned} Q(s) &:= \sum_{k=1}^{\infty} \frac{\Gamma\left(k+s\right)\Gamma\left(k-s\right) - \Gamma(k)^2}{\Gamma\left(k+\frac{1}{2}\right)^2} \\ &= \frac{8}{\pi}\int_0^{\pi/4} \frac{\left(\tan t\right)^{2\,s} + \left(\cot t\right)^{2\,s} - 2}{\cos 2t}\, t\,\mathrm{d}t \\ &= \frac{8}{\pi}\int_0^1 \frac{\left(x^s - x^{-s}\right)^2}{1-x^2}\, \arctan x\,\mathrm{d}x. \end{aligned}$$

The integrals above are obtained much as in the derivation of (69). For example,

$$Q\left(\frac{1}{4}\right) = \frac{8\,G}{\pi} - 4\,\log\left(1 + \frac{1}{\sqrt{2}}\right),$$

and there other nice evaluations. ◇

6 Closed forms at negative integers

We observe that (21) and (22) give analytic continuations which allow us to study negative moments. In [1] Adamchik studies such moments of K.

6.1 Negative moments

Adamchik's starting point is the study of $K_n = K_{n,0}$ for which Ramanujan appears to have known that

$$(2r+1)^2 K_{2r+1} - (2r)^2 K_{2r-1} = 1, \tag{70}$$

for $\Re r > -1/2$. For integer r this is a direct consequence of (25).

Experimentally, we found the following extension for general s by using integer relation methods with $s := 1/n$ $(3 \leq n \leq 9)$ to determine the coefficients:

$$\left((2r+1)^2 - 4s^2\right) K_{2r+1,s} - (2r)^2 K_{2r-1,s} = \cos(\pi s). \tag{71}$$

- For *integer* r this is established as follows.

Using (25) and the functional relation for the Gamma function, we have:

$$\begin{aligned}
&\left((2r+1)^2-4s^2\right)K_{2r+1,s}-4r^2K_{2r-1,s}\\
=\;&\frac{\pi\,(r!)^2}{\Gamma(\frac{1}{2}+r-s)\Gamma(\frac{1}{2}+r+s)}\left\{\sum_{k=0}^{r}\frac{(\frac{1}{2}-s)_k(\frac{1}{2}+s)_k}{(k!)^2}-\sum_{k=0}^{r-1}\frac{(\frac{1}{2}-s)_k(\frac{1}{2}+s)_k}{(k!)^2}\right\}\\
=\;&\frac{\pi\,(r!)^2}{\Gamma(\frac{1}{2}+r-s)\Gamma(\frac{1}{2}+r+s)}\frac{(\frac{1}{2}-s)_r(\frac{1}{2}+s)_r}{(r!)^2}\\
=\;&\frac{\pi}{\Gamma(\frac{1}{2}-s)\Gamma(\frac{1}{2}+s)}=\cos\pi s.
\end{aligned}$$

From (71) by *creative telescoping* one again deduces

$$K_{2n+1,s}=\frac{\cos\pi s}{4}\frac{n!^2}{\Gamma\left(n+\frac{3}{2}+s\right)\Gamma\left(n+\frac{3}{2}-s\right)}\sum_{k=0}^{n}\frac{\Gamma\left(k+\frac{1}{2}+s\right)\Gamma\left(k+\frac{1}{2}-s\right)}{k!^2}. \tag{72}$$

This provides another proof of Theorem 3.

Equation (13), when combined with (71), implies

$$E_{n,s}=\frac{(2s+1)^2K_{n,s}+\cos(\pi s)}{(2s+1)(2s+n+2)}, \tag{73}$$

which extends (17).

Adamchik also develops a reflection formula which in our terms is

$$K^*_{-1-2r} + K_{2r} = -\frac{\pi}{4^{2r}}\binom{2r}{r}^2 \{\log 2 + H_r - H_{2r}\} \tag{74}$$

for $r = 0, 1, 2, \ldots$.

Here

$$K^*_{-1-2r} \quad := \quad \lim_{t\to r}\left\{K_{-1-2t} - \frac{\binom{2n}{n}^2}{4^{2n+1}}\frac{\pi}{t-r}\right\}. \tag{75}$$

Note that, as examined in Theorem 9 of the next subsection, K^*_{-2r-1} removes the singularity at $-2r-1$.

- Hence, for $r = 0, 1, 2, \ldots$, K^*_{-1-2r} can be written as an infinite sum [1].

- What is the right s-generalization of (74)?

Example 8 (Terminating sums) While studying [1] we found the following results.

1. For $0 < a \leq 1$

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},a\\ 1,1+a\end{matrix}\middle|1\right) = \frac{4a}{\pi}\,{}_3F_2\left(\begin{matrix}1,1,1-a\\ \frac{3}{2},\frac{3}{2}\end{matrix}\middle|1\right). \tag{76} $$

   In particular when $a = 1/2$ then

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,\frac{3}{2}\end{matrix}\middle|1\right) = \frac{2}{\pi}\,{}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{3}{2},\frac{3}{2}\end{matrix}\middle|1\right) = \frac{4}{\pi}G. \tag{77} $$

$$ {}_3F_2\left(\begin{matrix}\frac{3}{4},1,1\\ \frac{3}{2},\frac{3}{2}\end{matrix}\middle|1\right) = \frac{\Gamma^4(1/4)}{16\pi}. \tag{78} $$

2. Moreover, for $n = 1, 2, 3, \ldots$

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},n\\ 1,1+n\end{matrix}\middle|1\right). \tag{79} $$

   always terminates. For example,

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},1\\ 1,2\end{matrix}\middle|1\right) = \frac{4}{\pi}, \qquad {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},2\\ 1,3\end{matrix}\middle|1\right) = \frac{40}{9\pi}. \tag{80} $$

3. Also for $n = 1, 2, \dots$

$$(2n+1)^2 \, {}_3F_2\left(\begin{matrix} 1, 1, -n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) \; - \; 4\,n^2 \, {}_3F_2\left(\begin{matrix} 1, 1, 1-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) = 1, \tag{81}$$

$$ {}_3F_2\left(\begin{matrix} 1, 1, 1-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) \; = \; \frac{4^{2n-1}}{n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{4^{2k}}. \tag{82}$$

and

$$ {}_3F_2\left(\begin{matrix} 1, 1, \frac{1}{2}-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) = \frac{\binom{2n}{n}^2}{4^{2n}} \left\{ 2G + \sum_{k=0}^{n-1} \frac{4^{2k}}{\binom{2k}{k}^2 (2k+1)^2} \right\} \tag{83}$$

4. For $0 < a \le 1$ and $n = 1, 2, \dots$

$$ {}_3F_2\left(\begin{matrix} 1, 1, 1-n-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) \; = \; \frac{(a)_n^2}{(a+\frac{1}{2})_n^2} \left\{ {}_3F_2\left(\begin{matrix} 1, 1, 1-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) + \frac{1}{4\,a^2} \sum_{k=0}^{n-1} \frac{(a+\frac{1}{2})_k^2}{(a+1)_k^2} \right\}. \tag{84}$$

and

$$ {}_3F_2\left(\begin{matrix} 1, 1, -a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) \; = \; \left(\frac{2a}{2a+1}\right)^2 \, {}_3F_2\left(\begin{matrix} 1, 1, 1-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1\right) + \frac{1}{(2a+1)^2}. \tag{85}$$

◇

6.2 Analyticity of $K_{\cdot,s}$ for $0 \le s < 1/2$

The analytic structure of $r \mapsto K_{r,s}$ is similar qualitatively for all values of s. This is illustrated in Figure 1 for $s = 1/3$ and $s = 1/\pi$ both superimposed on $s = 0$ (red). In all cases there are simple poles at odd negative integers with computable residues.

Theorem 9 (Poles of $K_{\cdot,s}$) *Let $R_{n,s}$ denote the residue of $K_{\cdot,s}$ at $r = -2n+1$. Then*

$$(a)\ R_{n+1,s} = \frac{\left(n - \frac{1}{2}\right)^2 - s^2}{n^2} R_{n,s}, \qquad (b)\ R_{1,s} = \frac{\pi}{2}. \tag{86}$$

Explicitly

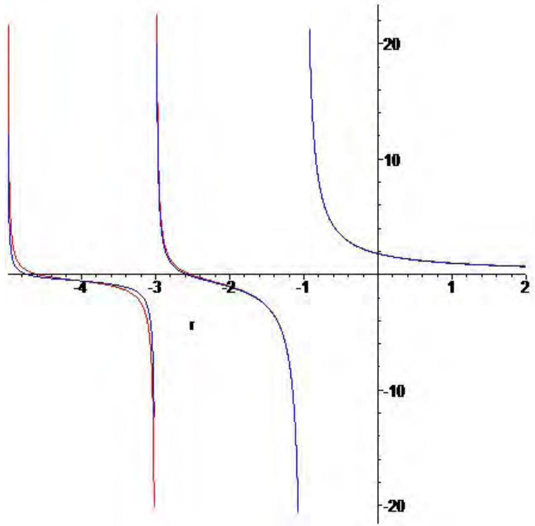
$$(a)\ R_{n,s} \ = \ \frac{\cos(\pi s)\Gamma\left(n - \frac{1}{2} + s\right)\Gamma\left(n - \frac{1}{2} - s\right)}{2\,\Gamma^2(n)}. \tag{87}$$

Proof. Recursion (86, a) follows from multiplying (71) by $2(r+n) = (2r+1) - (1-2n) = (2r-1) - (-2n-1)$ and computing the limits as $r \to -n$.

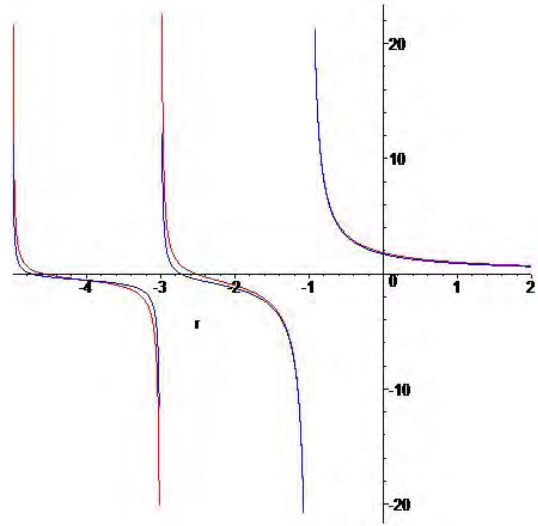
Directly from Theorem 2, we have the

$$R_{1,s} = \frac{\pi}{2} \lim_{r \to -1} \frac{r+1}{r+1}\, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2} - s, \frac{1}{2} + s, \frac{r+1}{2} \\ 1, \frac{r+3}{2} \end{matrix} \right| 1 \right) = \frac{\pi}{2},$$

which is (b); part (c) follows easily as a telescoping product. □



(a) $s = 0, 1/\pi$



(b) $s = 0, 1/3$

Figure 1: $r \mapsto K_{r,s}$ analytically continued to the real line.

6.3 Other rational values of s

Generally, directly integrating (1) or appealing to Theorem 2 yields the Saalschützian evaluation:

$$K_{(-1/2),s} = \pi\, {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}+s, \frac{1}{2}-s, \frac{1}{4} \\ 1, \frac{5}{4} \end{matrix}\right| 1\right). \tag{88}$$

For $s = 0$ only, $K_{-1/2,s}$ reduces to a case of Dixon's theorem [11, (2.3.3.5)] and yields

$$K_{(-1/2),0} = \frac{\Gamma\left(\frac{1}{4}\right)^4}{16\,\pi}, \tag{89}$$

a result known to Ramanujan.

Indeed, the two relevant specializations of Dixon's theorem are

$${}_3F_2\left(\left.\begin{matrix} \frac{1}{2}+s, \frac{1}{2}-s, \frac{1}{4} \\ 1-2s, \frac{5}{4}\,s-1 \end{matrix}\right| 1\right) = \frac{\Gamma\left(\frac{5}{4}-\frac{1}{2}\,s\right)\Gamma\left(\frac{1}{2}-\frac{3}{2}\,s\right)\Gamma\left(1-2\,s\right)\Gamma\left(\frac{5}{4}-s\right)}{\Gamma\left(\frac{3}{2}-s\right)\Gamma\left(\frac{3}{4}-2\,s\right)\Gamma\left(\frac{3}{4}-\frac{3}{2}\,s\right)\Gamma\left(1-\frac{1}{2}\,s\right)}$$

and more pleasingly,

$${}_3F_2\left(\left.\begin{matrix} \frac{1}{4}, \frac{1}{2}-s, \frac{1}{2}+s \\ \frac{3}{4}+s, \frac{3}{4}-s \end{matrix}\right| 1\right) = \frac{\sqrt{2}\,\pi}{\Gamma^2\left(\frac{5}{8}\right)}\,\frac{\Gamma\left(\frac{3}{4}+s\right)\Gamma\left(\frac{3}{4}-s\right)}{\Gamma\left(\frac{5}{8}+s\right)\Gamma\left(\frac{5}{8}-s\right)}.$$

We should like to be able to evaluate $K_{-1/3,1/6}$ and $K'_{-1/3,1/6}$ or equivalently

$$H_0 := \frac{\pi}{2}\int_0^1 {}_2F_1\left(\left.\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix}\right| t^3\right)\, \mathrm{d}t \quad \text{and} \quad H_0^* := \frac{\pi}{2}\int_0^1 {}_2F_1\left(\left.\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix}\right| 1-t^3\right)\, \mathrm{d}t, \tag{90}$$

respectively. So far we have met with partial success, see (91) and (93) below.

6.4 Moments with respect to t^3 instead

To evaluate H_0^* we first write

$$H_0^* = \frac{\pi}{6}\int_0^1 x^{-\frac{2}{3}}\,{}_2F_1\left(\begin{matrix}\frac{\pi}{6},\frac{2}{3}\\ 1\end{matrix}\,\middle|\,1-x\right)\mathrm{d}x = \frac{\pi}{6}\int_0^1 (1-x)^{-\frac{2}{3}}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\,\middle|\,x\right)\mathrm{d}x.$$

Now the integral (23) shows this is $\frac{\pi}{2}\,{}_3F_2\left(\begin{matrix}\frac{1}{3},\frac{2}{3},1\\ \frac{2}{3},\frac{4}{3}\end{matrix}\,\middle|\,1\right) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},1\\ \frac{4}{3}\end{matrix}\,\middle|\,1\right)$. By Gauss' theorem we arrive at

$$H_0^* = \frac{\pi}{2}\,\frac{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} = \frac{\sqrt{3}}{12}\,\Gamma^3\left(\frac{1}{3}\right). \tag{91}$$

This also follows directly from the analytic continuation of the formula in (38) of Theorem 4. Similarly,

$$H_0 = \frac{\pi}{6}\int_0^1 x^{-\frac{2}{3}}\;{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\,\middle|\,x\right)\mathrm{d}x = \frac{\pi}{3}\,{}_3F_2\left(\begin{matrix}\frac{1}{3},\frac{1}{3},\frac{2}{3}\\ 1,\frac{4}{3}\end{matrix}\,\middle|\,1\right).$$

If we use Bailey's identity:

$${}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\,\middle|\,1\right) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)}\,\frac{\Gamma(s)}{\Gamma(b+s)\Gamma(c+s)}\,{}_3F_2\left(\begin{matrix}d-a,e-a,s\\ s+b,s+c\end{matrix}\,\middle|\,1\right)$$

for $s = d+e-a-b-c$, when $\mathrm{Re}(s>0)$, $\mathrm{Re}(a)>0$ [11, Eqn. (2.3.3.7)], this can be transformed to

$$H_0 = \frac{\pi}{6}\,\frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{16}\,{}_3F_2\left(\begin{matrix}1,1,1\\ \frac{5}{3},\frac{5}{3}\end{matrix}\,\middle|\,1\right).$$

Next, applying (16.4.11) in the *Digital Library of Math Functions*

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}\middle| 1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} {}_3F_2\left(\begin{matrix} a, d-b, d-c \\ d, d+e-b-c \end{matrix}\middle| 1\right), $$

we arrive at

$$ H_0 = \frac{\pi}{6} \frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{4} \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n} \frac{3k-1}{3k+1}}{3n+2}, \tag{92} $$

while

$$ G = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n} \frac{1-2k}{1+2k}}{2n+1}. $$

- A current project is to partially automate this sort of hunt for *good* ${}_pF_q$ transformations (and validation).

Finally, we also arrive at a reworking of equation (45):

$$ 3 \sum_{k=0}^{\infty} (-2)^n H_n = 3 H_0^* = \frac{\sqrt{3}}{4} \Gamma^3 \left(\frac{1}{3}\right), \tag{93} $$

as a companion to (44).

7 Conclusion and open questions

Another impetus for this study was a query from Roberto Tauraso regarding whether, for integer $m = 0, 1, 2, \ldots$, one can find closed forms for

$$T(m,s) \quad := \quad \sum_{k=1}^{\infty} \frac{(\frac{1}{2}+s)_k\,(\frac{1}{2}-s)_k}{(1)_k^2}\,\frac{1}{k^m}. \tag{94}$$

We are able to write, more generally, that

$$T(m,s,\alpha) \quad := \quad \sum_{k=1}^{\infty} \frac{(\frac{1}{2}+s)_k\,(\frac{1}{2}-s)_k}{(1)_k^2}\,\frac{1}{(k+\alpha)^m} \tag{95}$$

$$= \quad \frac{\frac{1}{4}-s^2}{(\alpha+1)^m}\,{}_{m+2}F_{m+1}\left(\left.{\frac{3}{2}+s, \frac{3}{2}-s, \alpha+1, \cdots, \alpha+1 \atop 2, \alpha+2, \cdots, \alpha+2}\right|1\right). \tag{96}$$

- Sad to say, we have nothing better to provide than the hypergeometric form of (96).
- We should also very much like to know if one can evaluate the cubic moment $H_0 = \frac{2}{3}\,K_{-1/3,1/6}$ in (90) as we were able to do for $K_{-1/2,0}$. Both reduce to evaluation of cases of $\frac{\pi}{1+2s}\,{}_3F_2\left(\left.{\frac{1}{2}-s, \frac{1}{2}+s\ \frac{s}{2}+\frac{1}{4} \atop 1, \frac{s}{2}+\frac{5}{4}}\right|1\right)$ ($s = 0, 1/6$).
- Are there other non-trivial explicit fractional evaluations?
- What is the correct s-generalization of the reflection formula (75)?
- Finally, how do the connection results of (44), (93) generalize?

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