Colloquium on
Moments of Ramanujan’s generalized elliptic integrals
and extensions of Catalan’s constant

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1 Abstract

- We investigate the moments of Ramanujan’s alternative elliptic integrals and of related hypergeometric functions. This involved quite a lot of symbolic computation.

- Along the way we are able to give some surprising closed forms for Catalan-related constants and new hypergeometric identities.

- We also tell some of the striking history around these matters.
2 Introduction and background

As in [6, pp.178-179] for \(0 \leq s < 1/2, \ 0 \leq k \leq 1\), let

\[
K^s(k) := \frac{\pi}{2} \genfrac{\{}{\}}{0pt}{}{\frac{1}{2} - s, \frac{1}{2} + s}{1}{k^2}
\]  \(1\)

and

\[
E^s(k) := \frac{\pi}{2} \genfrac{\{}{\}}{0pt}{}{-\frac{1}{2} - s, \frac{1}{2} + s}{1}{k^2}
\]  \(2\)

We use the standard notation for hypergeometric functions, namely

\[
\genfrac{\{}{\}}{0pt}{}{a,b}{c}{z} := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}
\]

where \((a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1) \cdots (a+n-1)\) is the rising factorial or Pochhammer symbol; so \((1)_n = n!\).

Likewise

\[
\genfrac{\{}{\}}{0pt}{}{a,b,c}{d,e}{z} := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{z^n}{n!}
\]

We are interested in the moments given by

\[
K_n = K_{n,s} := \int_0^1 k^n K^s(k) \, dk, \quad E_n = E_{n,s} := \int_0^1 k^n E^s(k) \, dk
\]  \(3\)

for both integer and real values of \(n\).
Note that $K^s = K(-s)$. Euler’s transform [3, (2.2.7) and a contiguous relation give:

$$E^{(-s)} = \frac{4s(1-k^2)}{2s-1} K^s + \frac{2s+1}{2s-1} E^s.$$

The corresponding integral form of $K^s$ (due to Euler) is

$$K^s(k) = \frac{\cos(\pi s)}{2} \int_0^1 \frac{t^{s-1/2}}{(1-t)^{1/2+s}(1-k^2 t)^{1/2-s}} dt = \cos(\pi s) \int_0^{\pi/2} \frac{\tan^{2s}(\theta)}{(1-k^2 \sin^2 \theta)^{1/2-s}} d\theta.$$

- The latter has the nice feature of looking like the cleanest classical definition when $s = 0$: in which case $K$ gives the period of a pendulum and $E$ gives the arclength of an ellipse (hence the name) [6, Ch. 1].

- Many more forms for $K^s, E^s$ can be obtained from [http://dlmf.nist.gov/15.6](http://dlmf.nist.gov/15.6).

A key early result, due to Gauss (1812), when $\text{Re}(c-a-b) > 0$ is the closed form

$$2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$  

- For $3F_2$’s such closed forms have been intensely studied and evaluations are the exception not the rule.
There are four values for which these integrals are truly special:

\[ s \in \Omega := \left\{ 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3} \right\}. \]

That is, when \( \cos^2(\pi s) \) is rational.

- These are Ramanujan’s *alternative elliptic integrals* as displayed in [10] and first decoded in [6].
  A comprehensive study is given in [4]. (See also [9] and [2].)
- These four cases are precisely those which produce modular functions [6, §5.5].
- Their study is currently experiencing a renewal of interest, especially regarding related elliptic series for \( 1/\pi \) [5], [6, §5.5] and [7]. For example, Ramanujan’s \( (s = 0) \) series

\[
\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(2n)^3 42n + 5}{n 2^{12n+4}}.
\]

(7)

allows the computation of the second half of a binary digit-string to be computed without the first half!
2.1 Reciprocal series for $\pi$

Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910 [5, 6]. The most famous, with $s = 1/4$ is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{(k!)^4} \frac{(1103 + 26390k)}{3964k}.$$  \hspace{1cm} (8)

- Each term of (8) adds eight correct digits. Gosper used (8) for the computation of a then-record 17 million digits of $\pi$ in 1985—thereby completing my first proof of (8) [6, Ch. 3].

Shortly thereafter, David and Gregory Chudnovsky found the following variant, which uses $s = 1/3$ and lies in the quadratic number field $Q(\sqrt{-163})$ rather than $Q(\sqrt{58})$:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(3k)! (k!)^3} \frac{(13591409 + 545140134k)}{640320^{3k+3/2}}.$$  \hspace{1cm} (9)

- Each term of (9) adds 14 correct digits. The brothers used this formula several times, culminating in a 1994 calculation of $\pi$ to over four billion decimal digits. Their extraordinary story was told in a prizewinning New Yorker article on The Mountains of Pi by Richard Preston.
• In late 2009, equation (9) was used again for the then record computation of $\pi$ to 2.7 trillion places. In consequence, Fabrice Bellard has provided access to two trillion-digit integers whose ratio is bizarrely close to $\pi$.

• On August 6th, 2010 Shigeru Kondo and Alex Yee announced a new record of 5,000,000,000,000 decimal digits computed on Kondo’s home-built $18,000$ machine with $20$ hard disks in $90$ days. See [www.numberworld.org/misc_runs/pi-5t/details.html](http://www.numberworld.org/misc_runs/pi-5t/details.html).

• They used (9) with confirmation in 64 hours via the BBP algorithm for $\pi$ based on the formula

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right),$$

which can compute a string of digits in hex starting at say the five trillion digit mark

– For the whole story come to my public lecture on *Life of Pi* in Brisbane at the AustMS meeting on September 28.
2.2 Classical results

The coupling equation between $E^s$ and $K^s$ is given in [6, p. 178] and can be derived from the generalized hypergeometric differential equation (see http://dlmf.nist.gov/15.10). It is

$$E^s = (1 - k^2) K^s + \frac{k(1 - k^2)}{1 + 2s} \frac{d}{dk} K^s. \tag{11}$$

Integrating this by parts leads to

$$K_{2,s} = \frac{(1 + 2s) E_{0,s} - 2s K_{0,s}}{2 - 2s}. \tag{12}$$

In the same fashion, multiplying by $k^n$ before integrating the coupling provides a recursion for $K_{n+2,s}$:

$$K_{n+2,s} = \frac{(n - 2s) K_{n,s} + (1 + 2s) E_{n,s}}{n + 2(1 - s)}. \tag{13}$$

We also consider the complementary integrals:

$$K'_{s}(k) := K^s(\sqrt{1-k^2}) \quad \text{and} \quad E'_{s}(k) := E^s(\sqrt{1-k^2}).$$

The four integrals then satisfy a version of Legendre’s identity

$$E^s K'^{s} + K^s E'^{s} - K^s K'^{s} = \frac{\pi}{2} \frac{\cos(\pi s)}{1 + 2s}$$

for all $0 \leq k \leq 1$. 
In [6, pp. 188-89] the moments are determined for the classical case of $s = 0$ which give the original complete elliptic integrals $K, E$. These are linked by the equations (see [6, p. 9])

$$E = (1 - k^2) K + k(1 - k^2) \frac{dK}{dk}, \quad (15)$$

which is (11) with $s = 0$ and

$$E = K + k \frac{dE}{dk}, \quad (16)$$

from which we derive the following recursions:

**Theorem 1 (s=0)** For $n = 0, 1, 2, \ldots$

(a) $K_{n+2} = \frac{nK_n + E_n}{n + 2}$ and (b) $E_n = \frac{K_n + 1}{n + 2}$. \quad (17)

The recursion holds for real $n$.

Moreover,

$$
\begin{align*}
K_0 &= 2G, & K_1 &= 1, \\
E_0 &= G + \frac{1}{2}, & E_1 &= \frac{2}{3}.
\end{align*}
\quad (18)
$$
Here
\[ G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2} = L_{-4}(2) \]
is the Catalan constant whose irrationality is still not proven.

- This ignorance is part of my motivation for the study.

The current record for computation is **31.026 billion digits** in 2009.

Computations often use the following central binomial formula due to Ramanujan [6, last formula] or its recent generalizations:

\[
\frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n + 1)^2} + \frac{\pi}{8} \log(2 + \sqrt{3}) = G.
\]  \hspace{1cm} (20)

- PSLQ shows that if \( G \) is rational it has an enormous denominator.

- Amusingly in [1] Adamchik uses the moment evaluation above as a definition of \( G \)!

- We shall explore various ways to obtain the initial values.

- One may also profitably study fractional moments, see below and [1].
3 Basic results

We commence with various fundamental representations.

3.1 Hypergeometric closed forms

A concise closed form for the moments is

**Theorem 2 (Hypergeometric forms) For** $0 \leq s < \frac{1}{2}$ **we have**

$$K_{n,s} = \frac{\pi}{2(n+1)} {}_3F_2\left(\begin{array}{c}
\frac{1}{2} - s, \frac{1}{2} + s, \frac{n+1}{2} \\
1, \frac{n+3}{2}
\end{array} \middle| 1\right), \quad (21)$$

$$E_{n,s} = \frac{\pi}{2(n+1)} {}_3F_2\left(\begin{array}{c}
-\frac{1}{2} - s, \frac{1}{2} + s, \frac{n+1}{2} \\
1, \frac{n+3}{2}
\end{array} \middle| 1\right). \quad (22)$$

*These hold in the limit for* $s = \frac{1}{2}$.

**Proof.** To establish (21) and (22), we begin with

$$
\int_0^1 x^{u-1} (1-x)^{v-1} \binom{a, 1-a}{b} dx = \sum_{n=0}^{\infty} \frac{(a)_n (1-a)_n}{(b)_n n!} \int_0^1 x^{n+u-1} (1-x)^{v-1} dx \\
\quad = \sum_{n=0}^{\infty} \frac{(a)_n (1-a)_n (u)_n}{(b)_n (u+v)_n n!} \frac{\Gamma(u) \Gamma(v)}{\Gamma(u+v)} \\
\quad = \frac{\Gamma(u) \Gamma(v)}{\Gamma(u+v)} {}_3F_2\left(\begin{array}{c}
a, 1-a, u \\
b, u+v
\end{array} \middle| 1\right). \quad (23)
$$

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Similarly,
\[
\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} {_3}F_2\left(\begin{array}{c} a, -a, u \\ b, u + v \end{array} \right) = \int_0^1 x^{u-1}(1-x)^{v-1} \binom{a}{b} x \, dx.
\]

By applying these to (1) and (2) we immediately get (21) and (22).

\[\square\]

- As long as \(0 < s < 1/2\), the first series (21) is \textit{Saalschützian} [11]. That is, the denominator parameters add to one more than those in the numerator, but is not \textit{well poised}, and can be reduced to Gamma functions only for \(n = \pm 1\) (with \(n = -1\) a pole) since then it reduces to a \(2F_1\).

- The second (22) is not even Saalschützian, although it is \textit{nearly well poised} (whose definition [11] we do not need) and also can be reduced to Gamma functions for \(n = \pm 1\).

Thus, for \(|s| < 1/2\) we find

\[
K_{1,s} = \frac{\cos(\pi s)}{1 - 4s^2}, \quad E_{1,s} = \frac{2}{2s + 3} \frac{\cos(\pi s)}{1 - 4s^2}.
\] (24)

In general we obtain:
Theorem 3 (Odd moments) For odd integers $2m + 1$ and $m = 0, 1, 2, \ldots$, 

$$K_{2m+1,s} = \frac{\cos(\pi s) m!^2}{4 \Gamma\left(\frac{3}{2} - s + m\right) \Gamma\left(\frac{3}{2} + s + m\right)} \sum_{k=0}^{m} \frac{\Gamma\left(\frac{1}{2} - s + k\right) \Gamma\left(\frac{1}{2} + s + k\right)}{k!^2}.$$ (25)

Proof. In terms of the Legendre function,

$$2F_1\left(a, 1 - a \bigg| \frac{1}{z}\right) =: P_{-a}(1 - 2z),$$

where $y = P_\nu(x) = 2F_1\left(-\nu, \nu + 1 \bigg| \frac{1-x^2}{2}\right)$ solves $(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu + 1)y = 0$. In consequence we may deduce that

$$2F_1\left(a, 1 - a \bigg| \frac{1}{z}\right) = \frac{\sin \pi a}{\pi} \sum_{k=0}^{\infty} \frac{(a)_k(1-a)_k}{k!^2} (1 - z)^k \times \{2\Psi(1 + k) - \Psi(a + k) - \Psi(1 - a + k) - \log(1 - z)\},$$

where

$$\Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = \int_{0}^{\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt.$$ 

- Here $\gamma$ is Euler’s constant and $\Psi$ the digamma function (one of the workhorses of the special function world). Also

$\Psi(n+1) = H_n - \gamma$ where $H_n := 1 + 1/2 + \cdots + 1/n$. 

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Now, by integrating the series term-by-term and noting integral (23), we have

\[
\int_0^1 z^{n-1} _2F_1 \left( \begin{array}{c} a, 1-a \\ 1 \end{array} | z \right) \, dz = \frac{1}{n} \frac{3F_2}{1, n+1} \left( \begin{array}{c} a, 1-a, n \\ 1 \end{array} | 1 \right) = \frac{1}{n} \frac{(n-1)! \sin(\pi a)}{\pi} \sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{k!(k+n)!} \times \left\{ \Psi(1+k) + \Psi(n+1+k) - \Psi(a+k) - \Psi(1-a+k) \right\}.
\]

We note in passing that this offers an apparently new approach for summing this class of hypergeometric series; we exploit (23) again in section 6.4.

Thus, for example, by creative telescoping, one finds for any positive integer \( n \) that

\[
_3F_2 \left( \begin{array}{c} a, 1-a, n \\ 1, n+1 \end{array} | 1 \right) = \frac{\Gamma(n) \Gamma(1+n)}{\Gamma(a+n) \Gamma(1-a+n)} \sum_{k=0}^{n-1} \frac{(a)_k (1-a)_k}{k!^2}.
\]

(26)

Now, with \( n = m + 1 \) in (26) we conclude the proof of Theorem 3. \( \square \)

Similarly,

\[
_2F_1 \left( \begin{array}{c} a, -a \\ 1 \end{array} | z \right) = \frac{\sin(\pi a)}{\pi a} \left\{ 1 - a^2 \sum_{k=0}^{\infty} \frac{(a+1)_k (1-a)_k}{k!(k+1)!} (1-z)^{k+1} \times \left[ \Psi(a+1+k) + \Psi(1-a+k) - \Psi(k+1) - \Psi(k+2) + \ln(1-z) \right] \right\}.
\]

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For $m = 0$, Theorem 3 reduces to the evaluation given in (24). In general, it gives $\cos(\pi s)$ times a rational function. An equivalent, rather pretty, partial fraction decomposition is

$$K_{2m+1,s} = \frac{\cos \pi s}{2} \sum_{k=0}^{m} \frac{m!^2}{(m-k)!(m+k+1)!} \left( \frac{1}{2k + 1 - 2s} + \frac{1}{2k + 1 + 2s} \right).$$

(27)

This can easily be confirmed inductively, using say (71).

- For $s = 0$ this result originates with Ramanujan.
- Victor Adamchik [1] reprises its substantial history and extensions which include a formula due independently to Bailey and Hodgkinson in 1931 which subsumes (26).

It is

$$\text{3F}_2 \left( \begin{array}{c}{a, b, c + 1} \\ {a + b + n} \end{array} \big| 1 \right) = \frac{\Gamma(n)\Gamma(a + b + n)}{\Gamma(a + n)\Gamma(b + n)} \sum_{k=0}^{n-1} \frac{(a)_k(b)_k}{(c)_k(1)_k}.$$  

(28)

- The elliptic case of $a = b = 1/2, c = 1$ was in Ramanujan’s first letter to Hardy.
Example 1 (Digamma consequences) For $0 < a < 1/2$, consequences are neatly given using: $\gamma(\nu) := \frac{1}{2} \{ \Psi (\frac{\nu+1}{2}) - \Psi (\frac{\nu}{2}) \}$. Moreover

\[
\begin{align*}
\gamma \left( \frac{1}{2} \right) &= \frac{\pi}{2}, \\
\gamma \left( \frac{1}{4} \right) &= \frac{\pi}{\sqrt{2}} - \sqrt{2} \log (\sqrt{2} - 1), \\
\gamma \left( \frac{1}{3} \right) &= \frac{\pi}{\sqrt{3}} + \log 2, \\
\gamma \left( \frac{1}{6} \right) &= \pi + \sqrt{3} \log (2 + \sqrt{3}).
\end{align*}
\]

\[
\sum_{k=0}^{\infty} \frac{(a)_k(1-a)_k}{\left(\frac{3}{2}\right)_k k!} \left[ \Psi(k+1) + \Psi \left( k + \frac{3}{2} \right) - \Psi(k+a) - \Psi(k+1-a) \right] = \frac{2\gamma(a) - \pi \csc(\pi a)}{1 - 2a}.
\]

This in turn gives

\[
\begin{align*}
_3F_2 \left( a, 1-a, \frac{1}{2} \left| \begin{array}{c}
1, \frac{3}{2} \\
1
\end{array} \right. \right) &= \frac{2 \sin(\pi a)}{\pi (1 - 2a)} \gamma(a) - \frac{1}{1 - 2a}.
\end{align*}
\]

(29)

Taking the limit as $a \to 1/2$ in (29) gives two useful specializations:

\[
\begin{align*}
(a) \quad _3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| \begin{array}{c}
1, \frac{3}{2} \\
1
\end{array} \right. \right) &= \frac{4G}{\pi}, \\
(b) \quad \Psi' \left( \frac{1}{4} \right) &= \pi^2 + 8G,
\end{align*}
\]

(30) (31)

with (a) being known and very useful but far from obvious. \hfill \Diamond
The corresponding form for $E_{2m+1,s}$ is similar but less satisfactory:

$$E_{2m+1,s} = \frac{\pi}{4(m+1)} \frac{1}{\Gamma(\frac{3}{2}+s)\Gamma(\frac{1}{2}-s)} + \frac{\pi}{4} \frac{m!}{\Gamma(\frac{1}{2}+s)\Gamma(-\frac{1}{2}-s)}$$

$$\times \sum_{k=0}^{\infty} \frac{(\frac{3}{2}+s)_k(\frac{1}{2}-s)_k}{k!(k+m+2)!} \left\{ \Psi \left( \frac{3}{2} + s + k \right) + \Psi \left( \frac{1}{2} - s + k \right) - \Psi (k+1) - \Psi (3+m+k) \right\}.$$ (32)

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**Homage to Lobachevsky**

- Can one do better?
Example 2 (Other special values) For each $s \neq 0$ there are also two special values of $r$ for which $K_{r,s}$ also reduce to a $\text{}_2F_1$. They are obtained by solving $r + 3/2 = 1/2 \pm s$.

This and similar calculations for $E_{n,s}$ yield

\begin{align}
K_{(-2\pm 2s),s} &= -\frac{\cos(\pi s)}{(1 \mp 2s)^2}, \quad (33) \\
E_{(-2\pm 2s),s} &= -\frac{2}{(1 + 2s)} \frac{\cos(\pi s)}{(1 - 2s)^2}, \quad (34) \\
E_{(-4\pm 2s),s} &= -\frac{2}{(1 + 2s)} \frac{\cos(\pi s)}{(3 + 2s)^2}. \quad (35)
\end{align}

The $r$-recursions given above in (13) for $K_{r,s}$ and below in equation (73) for $E_{r,s}$ extend this to values of $r + 2n$, for $n$ integral.
Example 3 (Alternative moment expansions) We also obtain

\[ K_{0,s} = \frac{\cos(\pi s)}{2} \sum_{n=0}^{\infty} \left( \frac{1}{2} + s \right)_n \frac{(\frac{1}{2} - s)_n}{n!(\frac{3}{2})_n} \left\{ \Psi(n+1) + \Psi\left(\frac{3}{2} + n\right) - \Psi\left(\frac{1}{2} + n + s\right) - \Psi\left(\frac{1}{2} + n - s\right) \right\}, \]

\[ E_{0,s} = \frac{\cos(\pi s)}{2s + 1} + \cos(\pi s) \frac{2s + 1}{6} \sum_{n=0}^{\infty} \frac{(\frac{3}{2} + s)_n (\frac{1}{2} - s)_n}{n!(\frac{5}{2})_n} \times \left\{ \Psi(n+1) + \Psi\left(\frac{5}{2} + n\right) - \Psi\left(\frac{3}{2} + n + s\right) - \Psi\left(\frac{1}{2} + n - s\right) \right\}. \]
3.1.1 Half-integer values of $s$

For $s = m + 1/2$, and $m, n = 0, 1, 2 \ldots$ we can obtaining a terminating representation

$$K_{n,m+1/2} = \frac{\pi}{2(n+1)} \, _3F_2 \left( \begin{array}c -m, m + 1, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{array} \bigg| 1 \right)$$

$$= (-1)^m \frac{\pi}{4} \frac{\Gamma^2 \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} - m \right) \Gamma \left( \frac{n+3}{2} + m \right)}, \quad (36)$$

and likewise

$$E_{n,m+1/2} = \frac{\pi}{2} \sum_{k=0}^{m+1} \frac{(-m-1)_k (m+1)_k}{(n+1+2k) k!^2}. \quad (37)$$

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3.2 The complementary integrals

By contrast the complementary integral moments are less recondite.

**Theorem 4 (Complementary moments)** For \( n = 0, 1, 2, \ldots \) and \( 0 \leq s < \frac{1}{2} \) we have

\[
\begin{align*}
K'_{n,s} & = \frac{\pi}{4} \frac{\Gamma^2 \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+2-2s}{2} \right) \Gamma \left( \frac{n+2+2s}{2} \right)} \\
E'_{n,s} & = \frac{\pi}{2(n+1)} \frac{\Gamma^2 \left( \frac{n+3}{2} \right)}{\Gamma \left( \frac{n+2-2s}{2} \right) \Gamma \left( \frac{n+4+2s}{2} \right)}.
\end{align*}
\]

(38) 

(39)

These hold in the limit for \( s = \frac{1}{2} \). In particular, recursively we obtain for all real \( n \):

\[
\begin{align*}
(a) \quad K'_{n+2,s} & = \frac{(n+1)^2}{(n+2)^2 - 4s^2} K'_{n,s} \\
(b) \quad E'_{n,s} & = \frac{n+1}{n+2+2s} K'_{n,s}.
\end{align*}
\]

(40)

where

\[
\begin{align*}
(c) \quad K'_{0,s} & = \frac{\pi}{4} \frac{\sin (\pi s)}{s} \\
(d) \quad K'_{1,s} & = \frac{\cos (\pi s)}{1 - 4s^2}.
\end{align*}
\]

(41)
Proof. To establish (38) we recall that

\[ K_s' = \frac{\pi}{2} \, _2F_1\left( \frac{1}{2} - s, \frac{1}{2} + s \left| 1 - k^2 \right. \right), \]  

(42)

and so

\[
K_{n,s}' = \frac{\pi}{2} \int_0^1 x^n \, _2F_1\left( \frac{1}{2} - s, \frac{1}{2} + s \left| 1 - x^2 \right. \right) \, dx \\
= \frac{\pi}{4} \int_0^1 x^{\frac{n+1}{2} - 1} \, _2F_1\left( \frac{1}{2} - s, \frac{1}{2} + s \left| 1 - x \right. \right) \, dx \\
= \frac{\pi}{4} \int_0^1 (1 - x)^{\frac{n+1}{2} - 1} \, _2F_1\left( \frac{1}{2} - s, \frac{1}{2} + s \left| x \right. \right) \, dx \\
= \frac{\pi}{2(n+1)} \, _3F_2\left( \frac{1}{2} - s, \frac{1}{2} + s, 1 \left| 1 \right. \right) \\
= \frac{\pi}{2(n+1)} \, _2F_1\left( \frac{1}{2} - s, \frac{1}{2} + s \left| \frac{n+3}{2} \right. \right), 
\]

which is summable, by Gauss’ formula (6), to the desired result.

The proof of (39) is similar; and the recursions follow. \( \square \)
Example 4 (Complementary closed forms) Thence, with \( s = 0 \) and \( n = 0,1 \) we recover

\[
K'_0 = \frac{\pi^2}{4}, \quad E'_0 = \frac{\pi^2}{8}, \quad K'_1 = 1, \quad E'_1 = \frac{2}{3},
\]
as discussed in [6, p.98]. Correspondingly

\[
\begin{align*}
K'_{0,1/6} & = \frac{3\pi}{4}, \quad K'_{1,1/6} = \frac{9\sqrt{3}}{16}, \quad E'_{0,1/6} = \frac{9\pi}{28}, \quad K'_{1,1/6} = \frac{27\sqrt{3}}{80}, \\
K'_{0,1/3} & = \frac{3\sqrt{3}\pi}{8}, \quad K'_{1,1/3} = \frac{9}{10}, \quad E'_{0,1/3} = \frac{9\sqrt{3}\pi}{64}, \quad E'_{1,1/3} = \frac{27}{55}.
\end{align*}
\]

We note that \( \pi \), not \( \pi^2 \) appears in these evaluations, since in (41)(c),

\[
\frac{\sin (\pi s)}{s} \to \pi
\]
as \( s \to 0 \).

\( \diamond \)
3.2.1 Connecting moments and complementary moments

We first remark that a comparison of Theorems 3 and 4 shows that for all \( s \) we have 
\[
K'_{1,s} = K_{1,s} \quad \text{and} \quad E'_{1,s} = E_{1,s}.
\]

The formula
\[
\int_0^1 K(k) \frac{dk}{1 + k} = \int_0^1 K\left(\frac{1 - h}{1 + h}\right) \frac{dh}{1 + h} = \frac{1}{2} \int_0^1 K'(k) \, dk,
\]
(43)
is recorded in [6, p. 199]. It is proven by using the quadratic transformation [6, Thm 1.2 (b), p. 12] for the second equality and a substitution for the first. This implies
\[
2 \sum_{n=0}^{\infty} (-1)^n K_n = \frac{\pi^2}{4} = K'_0,
\]
(44)
on appealing to Theorem 4.

The corresponding identity for \( s = 1/6 \) is best written
\[
\int_0^1 \! _2F_1\left(\frac{1}{3}, \frac{2}{3} \left| \frac{1}{1 - t^3}\right\right) \ dt = 3 \int_0^1 \! _2F_1\left(\frac{1}{3}, \frac{2}{3} \left| t^3\right\right) \frac{dt}{1 + 2t},
\]
(45)
which follows analogously from the cubic transformation [8, Eqn 2.1] and a change of variables. This is a beautiful counterpart to (43) especially when the latter is written in hypergeometric form:
\[
\int_0^1 \! _2F_1\left(\frac{1}{2}, \frac{1}{2} \left| 1 - k^2\right\right) \, dk = 2 \int_0^1 \! _2F_1\left(\frac{1}{2}, \frac{1}{2} \left| k^2\right\right) \frac{dk}{1 + k},
\]
(46)
We further evaluate equation (45) in (93) of section 6.4.

Additionally, [6, p. 188] outlines how to derive
\[
\int_0^1 \frac{K(k)}{\sqrt{1 - k^2}} \, dk = K \left( \frac{1}{\sqrt{2}} \right)^2.
\]

Using the same technique, we generalize this to
\[
\int_0^1 \frac{K^s(k)}{\sqrt{1 - k^2}} \, dk = K^s \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{\cos^2(\pi s)}{16\pi} \Gamma^2 \left( \frac{1}{4} + \frac{s}{2} \right) \Gamma^2 \left( \frac{1}{4} - \frac{s}{2} \right).
\]

Here we have used Gauss’ \( _2 F_1 \) summation theorem (6) for the evaluation
\[
K^s \left( \frac{1}{\sqrt{2}} \right) = \frac{\cos \pi s}{4} \beta \left( \frac{1}{4} + \frac{s}{2}, \frac{1}{4} - \frac{s}{2} \right).
\]

By the generalized Legendre identity (14), which simplifies as the complementary integrals coincide with the original ones at \( 1/\sqrt{2} \), we obtain
\[
E^s \left( \frac{1}{\sqrt{2}} \right) = \frac{K^s \left( \frac{1}{\sqrt{2}} \right)}{2} + \frac{\pi \cos(\pi s)}{4(2s + 1)K^s \left( \frac{1}{\sqrt{2}} \right)}.
\]
3.3 Analytic continuation of results

We finish this section by recalling a useful—almost illegally so—theorem:

**Theorem 5 (Carlson (1914))** Let $f$ be analytic in the right half-plane $\Re z \geq 0$ and of exponential type (meaning that $|f(z)| \leq Me^{c|z|}$ for some $M$ and $c$), with the additional requirement that

$$|f(z)| \leq Me^{d|z|}$$

for some $d < \pi$ on the imaginary axis $\Re z = 0$. If $f(k) = 0$ for $k = 0, 1, 2, \ldots$ then $f(z) = 0$ identically.

- $\sin(z)$ show that the growth condition is optimal.
- Carlson’s theorem [12, 5.81] allow us to prove that many of the results in this paper hold generally as soon as they hold for integer $n$.
- For example, equations (70) or (71) hold generally as soon as the integral cases hold: once we check growth on the imaginary axis which is easy for hypergeometric functions.
- This matter is discussed at some length in [3, Thm 2.8.1 and sequel] in which an elegant 1941 proof by Selberg is given for the case in which $f(x)$ is bounded for $\Re(z) \geq 0$. 

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4 Closed form initial-values for various $s$

Many results work for all $s$ but (as we have seen) a few others are more satisfactory when $s \in \Omega$—since these four $K^s$ are the only modular functions ([6, Prop 5.7], [8]) amongst the generalized elliptic integrals $K^s$.

Empirically, we discovered an algebraic relation

$$2(1 + s) E_{0,s} - (1 + 2s) K_{0,s} = \frac{\cos(\pi s)}{1 + 2s}. \quad (48)$$

Equivalently, we exhibit a parametric reciprocal series for $\pi$:

$$\frac{2 \cos \pi s}{(1 + 2s)\pi} = (2 + 2s) \binom{3}{1,\frac{3}{2}} F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{3}{2} + s, -\frac{1}{2}, -s \\ \frac{1}{2}, \frac{3}{2} \end{array} \right) - (1 + 2s) \binom{3}{1,\frac{3}{2}} F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{3}{2} + s, \frac{1}{2}, -s \\ \frac{1}{2}, \frac{3}{2} \end{array} \right). \quad (49)$$

On using (12) to eliminate $E_{0,s}$ in (48) this becomes

$$K_{2,s} = \frac{K_{0,s} + \cos(\pi s)}{4 - 4s^2} \quad (50)$$

which in turn is a special case of (71) with $r = \frac{1}{2}$—as is justified by Carlson’s Theorem 5—thus proving our observation.
• Hence, to resolve all integral values, for a given $s$ we are left looking for satisfactory representations only for $K_{0,s}$.

We write

$$G_s := \frac{1}{2} K_{0,s} = \frac{\pi}{4} \, _3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2} - s, \frac{1}{2} + s \\ 1, \frac{3}{2} \end{array} \right| 1 \right).$$

and call this the associated or \textit{generalized Catalan} constant.

• What is a generalization for $G_s$ of the central binomial formula (20) for $G$?

• For various reasons, the results for $s = 1/6$ are especially interesting. This is the case corresponding to the cubic AGM [8].

From (21) we obtain

$$K_{0,s} = \frac{\pi}{2} \, _3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2} - s, \frac{1}{2} + s \\ 1, \frac{3}{2} \end{array} \right| 1 \right) = \frac{\cos (\pi s)}{2} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{1}{2} + n + s \right) \Gamma \left( \frac{1}{2} + n - s \right)}{(2n+1)(n!)^2}$$

$$= \frac{\cos (\pi s)}{2} \sum_{n=0}^{\infty} \beta \left( n + \frac{1}{2} - s, n + \frac{1}{2} + s \right) \frac{2n}{2n+1}$$

$$= \frac{\cos (\pi s)}{4} \int_{0}^{\frac{1}{2}} \arcsin \left( \frac{2 \sqrt{t} - t^2}{t^{1+s} (1-t)^{1-s}} \right) \, dt$$

$$= \frac{\cos (\pi s)}{2} \int_{0}^{\frac{\pi}{2}} \left\{ \tan^{2s} \left( \frac{\theta}{2} \right) + \cot^{2s} \left( \frac{\theta}{2} \right) \right\} \frac{\theta}{\sin \theta} \, d\theta.$$  

(51)
1. This uses the definition directly, see also [6, Prop 5.6], to attain the first identity after writing the rising factorials in terms of the $\beta$-function:

$$\beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt.$$ 

2. We exchange integral and sum to arrive at the penultimate integral. Moving the integral to $[-1/2, 1/2]$ and then making various trig substitutions, we arrive at the final result in (51).

For example, we have

$$K_{0,0} = \int_0^{\pi/2} \frac{\theta}{\sin \theta} d\theta = 2G.$$ 

The final equality has various derivations [6, 1]. These include contour integration as explored in section 5.

- If we now make the trigonometric substitution $t = \tan(\theta/2)$ in (51), and integrate the two resulting terms separately, we arrive at a central result.
Theorem 6 (Generalized Catalan values for $0 \leq s \leq \frac{1}{2}$)

\[
K_{0,s} = \cos(\pi s) \int_0^1 (t^{2s-1} + t^{-2s-1}) \arctan t \, dt
\]

\[
= \frac{\cos(\pi s)}{8s} \left\{ \Psi \left( \frac{3 - 2s}{4} \right) + \Psi \left( \frac{1 + 2s}{4} \right) - \Psi \left( \frac{1 - 2s}{4} \right) - \Psi \left( \frac{3 + 2s}{4} \right) \right\}
\]  

(52)

\[
= \frac{\cos \pi s}{4s} \left\{ \Psi \left( \frac{s}{2} + \frac{1}{4} \right) - \Psi \left( \frac{s}{2} + \frac{3}{4} \right) \right\} + \frac{\pi}{4s} = 2 G_s.
\]  

(53)

- For $s = 0$, L'Hôpital's rule provides

\[
\frac{1}{2} K_{0,0} = \frac{1}{16} \Psi' \left( \frac{1}{4} \right) - \frac{1}{8} \Psi' \left( \frac{3}{4} \right)
\]

which is precisely $G$.

- The digamma expression in (52) simplifies entirely when $s \in \Omega$ to the forms originally discovered in the next section. We finally obtain complete evaluations for $s \in \Omega$ as was our goal.

Corollary 1 (Generalized Catalan values for $s$ in $\Omega$)

\[
G_0 = G, \quad G_{1/6} = \frac{3}{4} \sqrt{3} \log 2, \quad G_{1/4} = \log \left( 1 + \sqrt{2} \right), \quad G_{1/3} = \frac{3}{8} \sqrt{3} \log \left( 2 + \sqrt{3} \right).
\]  

(54)
Mathematica, which currently knows more about the Psi function than Maple does, can evaluate the integral in Theorem 6 symbolically for some $s$.

For example, if $s = 1/12$, after simplification we have the very nice expression:

$$G_{1/12} = 3 \left( \sqrt{3} + 1 \right) \left\{ \log \left( \sqrt{2} - 1 \right) + \frac{\sqrt{3}}{2} \log \left( \sqrt{3} + \sqrt{2} \right) \right\}.$$  

• More generally, the evaluation requires only knowledge of $\sin(\pi s/2)$, and hence we can determine which $s$ give a reduction to radicals.

As a last example:

$$G_{1/5} = \frac{5}{8} \sqrt{5 + 2 \sqrt{5}} \left\{ \frac{\sqrt{5} - 1}{2} \text{arcsinh} \left( \sqrt{5 + 2 \sqrt{5}} \right) - \text{arcsinh} \left( \sqrt{5 - 2 \sqrt{5}} \right) \right\}.$$  

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5 Contour integrals for $K_{0,s}$

By contour integration on the infinite rectangle above $[0, \pi/2]$ (see the Figure) we obtain

\begin{equation}
G_0 = \frac{1}{2} \int_{0}^{\infty} \frac{t}{\cosh t} \, dt = \int_{0}^{\infty} \frac{te^{-t}}{1 + e^{-2t}} \, dt = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^2} = G.
\end{equation}

- Here we used the geometric series, and integrated termwise the $\Gamma$-function terms we got.
- The final evaluation is definitional.
- Maple will evaluate $x \mapsto \int_{0}^{x} \frac{t}{\cosh t} \, dt$, in dilogarithmic terms. Care to simplify it?
• Contour integration over the rectangle (Cauchy) provides an integral for $G_s$ with $0 \leq s < 1/2$. Namely, we integrate:

1. From 0 to $\pi/2$.
2. From $\pi/2$ to $\pi/2 + N i\infty$ for large $N$.
3. From $\pi/2 + N i\infty$ to $0 + N i\infty$ and home.

We check the integral on the top of the rectangle goes to zero as $N \to \infty$ and deduce:

**Theorem 7 (Contour integral for $G_s$)** For $0 \leq s < 1/2$ we have

$$2G_s = K_{0,s} = 2^{2s} \sin (2\pi s) \int_0^\infty \frac{(\cosh t)^{4s} - (\sinh t)^{4s}}{(\sinh 2t)^{2s+1}} t \, dt$$

$$+ \cos (\pi s) \int_0^\infty \frac{\cos (2 s \arctan (\sinh t))}{\cosh t} \, t \, dt. \quad (56)$$

• When $s = 0$ this is the previous result.
• When $s = 1/2$ the result fails ($\pi/2 = 0$).
• When $s = 1/4$ it is especially simple ...
Example 5 (Experimentally obtained evaluations) For $s = 1/4$, equation (56) becomes

$$K_{0,1/4} = \sqrt{2} \int_0^\infty \frac{\cosh t - \sinh t}{(\sinh 2t)^{3/2}} t dt + 2 \sqrt{2} \int_0^\infty \frac{\cosh t}{(\cosh 2t)^{3/2}} t dt,$$

(57)

with numerical value $\approx 1.7627471740392$. Here for the first time the specific form of the root of unity has played a role.

Quite remarkably, when we—much as before—converted the integrand to exponential form and apply the binomial theorem we obtained $\Gamma$-function values which became:

$$G_{1/4} = \sum_{n=0}^{\infty} \left( -\frac{3}{2} \right) \frac{12n + 8n^2 + 5 + (-1)^n (2n + 1)^2}{8 (n + 1)^2 (2n + 1)^2} = \log \left( 1 + \sqrt{2} \right).$$

(58)

Having thus proven this, we then discovered using the integer relation algorithm $\text{PSLQ}$ and the Maple $\text{identify}$ function that:

$$K_{0,1/6} = \frac{3}{2} \sqrt{3} \log 2,$$

(59)

with numerical value $\approx 1.8008492007794$, and a similar evaluation:

$$K_{0,1/3} = \frac{3}{2} \sqrt{3} \log \left( 1 + \sqrt{3} \right) - \frac{3}{4} \sqrt{3} \log (2).$$

(60)

with numerical value $\approx 1.7107784916770$. □

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Example 6 (Further integrals) We have discovered additionally, using inverse symbolic computational methods (http://carma.newcastle.edu.au/isc2), that

\[
\int_0^\infty \frac{(\cosh t)^{4/3} - (\sinh t)^{4/3}}{(\sinh t \cosh t)^{5/3}} \, dt = \frac{9}{4} \log(3),
\]

and

\[
\int_0^\infty \frac{(\cosh t)^{2/3} - (\sinh t)^{2/3}}{(\sinh t \cosh t)^{4/3}} \, dt = \frac{3}{2} \log \left( \frac{27}{16} \right).
\]

In light of Corollary 1 the discoveries of these two examples are now proven.
5.1 Contour integral based series for $K_{0,s}$

Let us write

$$K_{0,s} = \sin (2\pi s) S(s) + \cos (\pi s) C(s)$$

(61)

where

$$S(s) := 2^{2s} \int_0^\infty \frac{(\cosh t)^{4s} - (\sinh t)^{4s}}{(\sinh 2t)^{2s+1}} t \, dt$$

(62)

$$C(s) := \int_0^\infty \frac{\cos (2s \arctan (\sinh t))}{\cosh t} t \, dt.$$ (63)

- To evaluate $S(s)$ we make a substitution $u = \tanh(t)$. We obtain

$$S(s) = \frac{1}{2} \int_0^1 (u^{-2s-1} - u^{2s-1}) \arctanh(u) \, du$$

$$= \frac{-1}{8s} \left( 2\gamma + 4 \log(2) + \Psi \left( \frac{1}{2} - s \right) + \Psi \left( \frac{1}{2} + s \right) \right).$$

(64)

- Here as above

$$\gamma := \lim_{n \to \infty} H_n - \log n$$

is the Euler-Mascheroni constant whose irrationality is both certain but not proven.
To evaluate $C(s)$ we note that
\[
\cos (2 s \arctan (\sinh t)) = \cos (2 s \arcsin (\tanh t)) = _2F_1 \left( \begin{array}{c} s, -s \\ \frac{1}{2} \end{array} \right| \tanh^2 t \right)
\]
and so we obtain a converging (finite if $s = 0$) series
\[
C(s) = \int_0^\infty \frac{\cos (2 s \arctan (\sinh t))}{\cosh t} t \, dt = \sum_{n=0}^{\infty} \frac{(s)_n (-s)_n \tau_n}{\left(\frac{1}{2}\right)_n n!}
\]
where
\[
\tau_n := \int_0^\infty \frac{x^{2n}}{(1 + x^2)^{n+1}} \arcsinh(x) \, dx,
\]
and where we have expanded termwise.

Moreover,
\[
\tau_{m+2} = \frac{(13 + 8 m^2 + 20 m) \tau_{m+1} - 2 (m + 1) (2 m + 1) \tau_m}{2 (m + 2) (2 m + 3)}
\]
where $\tau_0 = K_0 = 2G$ and $\tau_1 = E_0 = G + \frac{1}{2}$.

In particular $C(0) = 2G$. 

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A closed form for $\tau_n$ is easily obtained. It is

$$\tau_n = \beta \left( n + \frac{1}{2}, \frac{1}{2} \right) \left\{ \frac{2G}{\pi} + \frac{1}{4} \sum_{k=1}^{n} \frac{\Gamma(k)^2}{\Gamma \left( k + \frac{1}{2} \right)^2} \right\}. \quad (68)$$

Collecting up evaluations, we deduce that

$$K_{0,s} = \sin(2\pi s) \left\{ \frac{-1}{8s} \left( 2\gamma + 4 \log(2) + \Psi \left( \frac{1}{2} - s \right) + \Psi \left( \frac{1}{2} + s \right) \right) \right\}$$

$$+ \frac{\sin(2\pi s)}{\pi s} \left\{ G - \pi \sum_{k=0}^{\infty} \frac{\Gamma(k + s + 1) \Gamma(k - s + 1) - k!^2}{8 \Gamma \left( k + \frac{3}{2} \right)^2} \right\},$$

since on interchanging order of summation

$$\frac{\pi}{4} \cos(\pi s) \sum_{n=1}^{\infty} \frac{(s)_n (-s)_n}{n!^2} \sum_{k=1}^{n} \frac{\Gamma(k)^2}{\Gamma \left( k + \frac{1}{2} \right)^2} = -\frac{\sin 2\pi s}{8s} \sum_{k=1}^{\infty} \frac{\Gamma(k + s) \Gamma(k - s) - \Gamma(k)^2}{\Gamma \left( k + \frac{1}{2} \right)^2}.$$

This ultimately yields:

**Theorem 8 (Contour series for $G_s$)**

$$G_s = \frac{\sin(2\pi s)}{16s} \left( \sum_{k=1}^{\infty} \frac{\Gamma(k)^2 - \Gamma(k + s)\Gamma(k - s)}{\Gamma \left( k + \frac{1}{2} \right)^2} + 2\Psi \left( \frac{1}{2} \right) - 2\Psi \left( s + \frac{1}{2} \right) + \pi \tan(\pi s) + \frac{8G}{\pi} \right). \quad (69)$$
Example 7 (A related series) Note for $s = 0$ we obtain precisely $G_0 = G$ as all other terms are zero. Comparing, (69) to (52) leads to a closed form for the infinite series $Q(s)$ given by

$$Q(s) := \sum_{k=1}^{\infty} \frac{\Gamma(k+s)\Gamma(k-s) - \Gamma(k)^2}{\Gamma(k + \frac{1}{2})^2} \frac{\Gamma(k + \frac{1}{2})}{2 \Gamma(k + \frac{1}{2})^2} = \frac{8}{\pi} \int_{\pi/4}^{\pi} \frac{(\tan t)^2 + (\cot t)^2 - 2}{\cos 2t} t \, dt$$

$$= \frac{8}{\pi} \int_{0}^{1} \frac{(x^s - x^{-s})^2}{1 - x^2} \arctan x \, dx.$$

The integrals above are obtained much as in the derivation of (69). For example,

$$Q\left(\frac{1}{4}\right) = \frac{8G}{\pi} - 4 \log \left(1 + \frac{1}{\sqrt{2}}\right),$$

and there other nice evaluations. ☐

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6  Closed forms at negative integers

We observe that (21) and (22) give analytic continuations which allow us to study negative moments. In [1] Adamchik studies such moments of $K$.

6.1  Negative moments

Adamchik’s starting point is the study of $K_n = K_{n,0}$ for which Ramanujan appears to have known that

\[(2r + 1)^2 K_{2r+1} - (2r)^2 K_{2r-1} = 1, \tag{70}\]

for $\Re r > -1/2$. For integer $r$ this is a direct consequence of (25).

Experimentally, we found the following extension for general $s$ by using integer relation methods with $s := 1/n$ ($3 \leq n \leq 9$) to determine the coefficients:

\[(2r + 1)^2 - 4s^2) K_{2r+1,s} - (2r)^2 K_{2r-1,s} = \cos(\pi s). \tag{71}\]
• For integer $r$ this is established as follows.

Using (25) and the functional relation for the Gamma function, we have:

$$
\left( (2r + 1)^2 - 4s^2 \right) K_{2r+1,s} - 4r^2 K_{2r-1,s} = \pi \frac{(r!)^2}{\Gamma(\frac{1}{2} + r - s)\Gamma(\frac{1}{2} + r + s)} \left\{ \sum_{k=0}^{r} \frac{(\frac{1}{2} - s)_k(\frac{1}{2} + s)_k}{(k!)^2} - \sum_{k=0}^{r-1} \frac{(\frac{1}{2} - s)_k(\frac{1}{2} + s)_k}{(k!)^2} \right\}
$$

$$
= \pi \frac{(r!)^2}{\Gamma(\frac{1}{2} + r - s)\Gamma(\frac{1}{2} + r + s)} \frac{(\frac{1}{2} - s)_r(\frac{1}{2} + s)_r}{(r!)^2}
$$

$$
= \frac{\pi}{\Gamma(\frac{1}{2} - s)\Gamma(\frac{1}{2} + s)} = \cos \pi s.
$$

From (71) by creative telescoping one again deduces

$$
K_{2n+1,s} = \frac{\cos \pi s}{4} \frac{n!^2}{\Gamma(n + \frac{3}{2} + s)\Gamma(n + \frac{3}{2} - s)} \sum_{k=0}^{n} \frac{\Gamma(k + \frac{1}{2} + s)\Gamma(k + \frac{1}{2} - s)}{k!^2}. \quad (72)
$$

This provides another proof of Theorem 3.

Equation (13), when combined with (71), implies

$$
E_{n,s} = \frac{(2s + 1)^2 K_{n,s} + \cos(\pi s)}{(2s + 1)(2s + n + 2)}, \quad (73)
$$

which extends (17).
Adamchik also develops a reflection formula which in our terms is

\[
K_{-1-2r}^* + K_{2r} = -\frac{\pi}{4^{2r}} \left(\frac{2r}{r}\right)^2 \{\log 2 + H_r - H_{2r}\}
\]  

(74)

for \( r = 0, 1, 2, \ldots \).

Here

\[
K_{-1-2r}^* := \lim_{t \to r} \left\{ K_{-1-2t} - \frac{(2n)^2}{4^{2n+1}} \frac{\pi}{t-r} \right\}.
\]

(75)

Note that, as examined in Theorem 9 of the next subsection, \( K_{-2r-1}^* \) removes the singularity at \(-2r - 1\).

- Hence, for \( r = 0, 1, 2, \ldots \), \( K_{-1-2r}^* \) can be written as an infinite sum [1].

- What is the right s-generalization of (74)?

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Example 8 (Terminating sums) While studying [1] we found the following results.

1. For $0 < a \leq 1$
   \[
   3F_2 \left( \frac{1}{2}, \frac{1}{2}, a \left| \frac{1}{1, 1 + a} \right| \right) = \frac{4a}{\pi} 3F_2 \left( \frac{1, 1, 1 - a}{\frac{3}{2}, \frac{3}{2}} \left| \frac{1}{1} \right| \right). \tag{76}
   \]
   In particular when $a = 1/2$ then
   \[
   3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| \frac{1}{1, \frac{1}{2}} \right| \right) = \frac{2}{\pi} 3F_2 \left( \frac{1, 1, \frac{1}{2}}{\frac{3}{2}, \frac{3}{2}} \left| \frac{1}{1} \right| \right) = \frac{4}{\pi} G. \tag{77}
   \]
   \[
   3F_2 \left( \frac{3}{4}, 1, 1 \left| \frac{1}{1, \frac{3}{2}} \right| \right) = \frac{\Gamma^4(1/4)}{16\pi}. \tag{78}
   \]

2. Moreover, for $n = 1, 2, 3, \ldots$
   \[
   3F_2 \left( \frac{1}{2}, \frac{1}{2}, n \left| \frac{1}{1, 1 + n} \right| \right). \tag{79}
   \]
   always terminates. For example,
   \[
   3F_2 \left( \frac{1}{2}, \frac{1}{2}, 1 \left| \frac{1}{1, 2} \right| \right) = \frac{4}{\pi}, \quad 3F_2 \left( \frac{1}{2}, \frac{1}{2}, 2 \left| \frac{1}{1, 3} \right| \right) = \frac{40}{9\pi}. \tag{80}
   \]
3. Also for \( n = 1, 2, \ldots \)

\[
(2n + 1)^2 \, {}_3F_2 \left( \frac{1, 1, -n}{3/2, 3/2} \bigg| 1 \right) - 4n^2 \, {}_3F_2 \left( \frac{1, 1, 1 - n}{3/2, 3/2} \bigg| 1 \right) = 1, \quad (81)
\]

\[
{}_3F_2 \left( \frac{1, 1, 1 - n}{3/2, 3/2} \bigg| 1 \right) = \frac{4^{2n-1}}{n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{(2k)^2}{\binom{2k}{k}^2} \quad (82)
\]

and

\[
{}_3F_2 \left( \frac{1, 1, \frac{1}{2} - n}{3/2, 3/2} \bigg| 1 \right) = \frac{(2n)^2}{4^{2n}} \left\{ 2G + \sum_{k=0}^{n-1} \frac{4^{2k}}{\binom{2k}{k}^2 (2k + 1)^2} \right\} \quad (83)
\]

4. For \( 0 < a \leq 1 \) and \( n = 1, 2, \ldots \)

\[
{}_3F_2 \left( \frac{1, 1, 1 - n - a}{3/2, 3/2} \bigg| 1 \right) = \frac{(a)_{n}^2}{(a + 1/2)^n} \left\{ {}_3F_2 \left( \frac{1, 1, 1 - a}{3/2, 3/2} \bigg| 1 \right) + \frac{1}{4a^2} \sum_{k=0}^{n-1} \frac{(a + 1/2)^2}{(a + 1)^3} \right\} \quad (84)
\]

and

\[
{}_3F_2 \left( \frac{1, 1, -a}{3/2, 3/2} \bigg| 1 \right) = \left( \frac{2a}{2a + 1} \right)^2 {}_3F_2 \left( \frac{1, 1, 1 - a}{3/2, 3/2} \bigg| 1 \right) + \frac{1}{(2a + 1)^2} \quad (85)
\]

\( \diamond \)
6.2 Analyticity of $K_{r,s}$ for $0 \leq s < 1/2$

The analytic structure of $r \mapsto K_{r,s}$ is similar qualitatively for all values of $s$. This is illustrated in Figure 1 for $s = 1/3$ and $s = 1/\pi$ both superimposed on $s = 0$ (red). In all cases there are simple poles at odd negative integers with computable residues.

Theorem 9 (Poles of $K_{r,s}$) Let $R_{n,s}$ denote the residue of $K_{r,s}$ at $r = -2n + 1$. Then

\begin{equation}
(a) \quad R_{n+1,s} = \frac{(n - \frac{1}{2})^2 - s^2}{n^2} R_{n,s}, \quad (b) \quad R_{1,s} = \frac{\pi}{2}.
\end{equation}

\text{Explicitly}

\begin{equation}
(a) \quad R_{n,s} = \frac{\cos(\pi s) \Gamma(n - \frac{1}{2} + s) \Gamma(n - \frac{1}{2} - s)}{2 \Gamma^2(n)}.
\end{equation}

\textbf{Proof.} Recursion (86, a) follows from multiplying (71) by $2(r+n) = (2r+1) - (1-2n) = (2r-1) - (-2n-1)$ and computing the limits as $r \to -n$.

Directly from Theorem 2, we have the

\[
R_{1,s} = \frac{\pi}{2} \lim_{r \to -1} \frac{r + 1}{r + 1} {3 \choose 2} \left( \begin{array}{c}
\frac{1}{2} - s, \frac{1}{2} + s, \frac{r+1}{2} \\
\frac{r+3}{2}, 1
\end{array} \right) = \frac{\pi}{2},
\]

which is (b); part (c) follows easily as a telescoping product. \hfill \Box
Figure 1: $r \mapsto K_{r,s}$ analytically continued to the real line.
6.3 Other rational values of $s$

Generally, directly integrating (1) or appealing to Theorem 2 yields the Saalschützian evaluation:

$$K_{(-1/2),s} = \pi_3 F_2 \left( \frac{1}{2} + s, \frac{1}{2} - s, \frac{1}{4} \middle| 1 \right).$$

(88)

For $s = 0$ only, $K_{-1/2,s}$ reduces to a case of Dixon’s theorem [11, (2.3.3.5)] and yields

$$K_{(-1/2),0} = \frac{\Gamma \left( \frac{1}{4} \right)^4}{16 \pi},$$

(89)

a result known to Ramanujan.

Indeed, the two relevant specializations of Dixon’s theorem are

$$3 F_2 \left( \frac{1}{2} + s, \frac{1}{2} - s, \frac{1}{4} \middle| 1 \right) = \frac{\Gamma \left( \frac{5}{4} - s \right) \Gamma \left( \frac{1}{2} - \frac{3}{2} s \right) \Gamma \left( 1 - 2 s \right) \Gamma \left( \frac{5}{4} - \frac{1}{2} s \right)}{\Gamma \left( \frac{3}{2} - s \right) \Gamma \left( \frac{3}{2} - \frac{3}{2} s \right) \Gamma \left( 1 - \frac{1}{2} s \right)}$$

and more pleasingly,

$$3 F_2 \left( \frac{1}{4}, \frac{1}{2} - s, \frac{1}{2} + s \middle| 1 \right) = \frac{\sqrt{2} \pi \Gamma \left( \frac{3}{4} + s \right) \Gamma \left( \frac{3}{4} - s \right)}{\Gamma^2 \left( \frac{5}{8} + s \right) \Gamma \left( \frac{5}{8} - s \right)}.$$

We should like to be able to evaluate $K_{-1/3,1/6}$ and $K'_{-1/3,1/6}$ or equivalently

$$H_0 := \frac{\pi}{2} \int_0^1 2 F_1 \left( \frac{1}{3}, \frac{2}{3} \middle| t^3 \right) \, dt \quad \text{and} \quad H_0^* := \frac{\pi}{2} \int_0^1 2 F_1 \left( \frac{1}{3}, \frac{2}{3} \middle| 1 - t^3 \right) \, dt,$$

(90)

respectively. So far we have met with partial success, see (91) and (93) below.
6.4 Moments with respect to $t^3$ instead

To evaluate $H^*_0$ we first write

$$H^*_0 = \frac{\pi}{6} \int_0^1 x^{-\frac{2}{3}}_2 F_1 \left( \frac{\pi}{6}, \frac{2}{3} \left| \frac{1}{1} - x \right) dx = \frac{\pi}{6} \int_0^1 (1-x)^{-\frac{2}{3}}_2 F_1 \left( \frac{1}{3}, \frac{2}{3} \left| x \right) dx.$$

Now the integral (23) shows this is $\frac{\pi}{2} 3 F_2 \left( \frac{\frac{1}{3}, 2}{\frac{1}{3}, \frac{1}{3}} \left| 1 \right) = \frac{\pi}{2} 2 F_1 \left( \frac{1}{1}, \frac{1}{1} \left| 1 \right)$. By Gauss’ theorem we arrive at

$$H^*_0 = \frac{\pi}{2} \frac{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} = \frac{\sqrt{3}}{12} \Gamma(\frac{3}{2}).$$

This also follows directly from the analytic continuation of the formula in (38) of Theorem 4. Similarly,

$$H_0 = \frac{\pi}{6} \int_0^1 x^{-\frac{2}{3}}_2 F_1 \left( \frac{1}{3}, \frac{2}{3} \left| x \right) dx = \frac{\pi}{3} 3 F_2 \left( \frac{\frac{1}{3}, \frac{1}{3}, \frac{2}{3}}{1, \frac{1}{3}} \left| 1 \right).$$

If we use Bailey’s identity:

$$3 F_2 \left( a, b, c \left| d, e \right| 1 \right) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)} \frac{\Gamma(s)}{\Gamma(b+s)\Gamma(c+s)} 3 F_2 \left( d - a, e - a, s \left| s + b, s + c \right| 1 \right)$$

for $s = d + e - a - b - c$, when Re($s > 0$), Re($a > 0$) [11, Eqn. (2.3.3.7)], this can be transformed to

$$H_0 = \frac{\pi}{6} \frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{16} 3 F_2 \left( \frac{1, 1, 1}{\frac{5}{3}, \frac{5}{3}} \left| 1 \right).$$

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Next, applying (16.4.11) in the *Digital Library of Math Functions*

\[
3F_2 \left( \begin{array}{c} a, b, c \\ d, e \end{array} \left| 1 \right. \right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} 3F_2 \left( \begin{array}{c} a, d-b, d-c \\ d, d+e-b-c \left| 1 \right. \right),
\]

we arrive at

\[
H_0 = \frac{\pi}{6} \frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{4} \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n} \frac{3k-1}{3k+1}}{3n+2}, \tag{92}
\]

while

\[
G = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n} \frac{1-2k}{1+2k}}{2n+1}.
\]

- A current project is to partially automate this sort of hunt for good $pF_q$ transformations (and validation).

Finally, we also arrive at a reworking of equation (45):

\[
3 \sum_{k=0}^{\infty} (-2)^n H_n = 3 H_0^* = \frac{\sqrt{3}}{4} \Gamma^3 \left( \frac{1}{3} \right), \tag{93}
\]

as a companion to (44).
7 Conclusion and open questions

Another impetus for this study was a query from Roberto Tauraso regarding whether, for integer $m = 0, 1, 2, \ldots$, one can find closed forms for
\[
T(m, s) := \sum_{k=1}^{\infty} \frac{(\frac{1}{2} + s)_k (\frac{1}{2} - s)_k}{(1)_k^2} \frac{1}{k^m}.
\]  

We are able to write, more generally, that
\[
T(m, s, \alpha) := \sum_{k=1}^{\infty} \frac{(\frac{1}{2} + s)_k (\frac{1}{2} - s)_k}{(1)_k^2} \frac{1}{(k + \alpha)^m} 
= \frac{\frac{1}{4} - s^2}{(\alpha + 1)^m + 2} F_{m+1} \left( \begin{array}{c} \frac{3}{2} + s, \frac{3}{2} - s, \alpha + 1, \cdots, \alpha + 1 \\ \frac{\alpha + 2, \cdots, \alpha + 2}{} \end{array} \bigg| 1 \right). 
\]

\begin{itemize}
  \item Sad to say, we have nothing better to provide than the hypergeometric form of (96).
  \item We should also very much like to know if one can evaluate the cubic moment $H_0 = \frac{2}{3} K_{-1,3,1/6}$ in (90) as we were able to do for $K_{-1/2,0}$. Both reduce to evaluation of cases of $\pi \frac{1+2s}{1+2s} 3 F_2 \left( \begin{array}{c} \frac{1}{2} - s, \frac{1}{2} + s, \frac{1}{2} + 1 \\ \frac{1}{2} + s, \frac{1}{2} + \frac{1}{2} \end{array} \bigg| 1 \right)$ ( $s = 0, 1/6$).
  \item Are there other non-trivial explicit fractional evaluations?
  \item What is the correct $s$-generalization of the reflection formula (75)?
  \item Finally, how do the connection results of (44), (93) generalize?
\end{itemize}
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References


