A quick lesson on the Gelfand integral

\((\Omega, \Sigma, \mu)\) prob space \times Banach space

\(f : \Omega \to X^*\), \(f(\omega)(x) \in L^1(\mu)\) \(\forall x\).

Easy: \(x \mapsto f(\omega)(x)\) has a closed graph.

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Example: \(f : \Omega \to X^*\) is weak\-\* continuous. Then \(f\) exists.
$u : X \to Y$ absolutely $p$-summing ($1 \leq p < \infty$)
if $\exists C > 0$ so that given $x_1, \ldots, x_n \in X$

$$\sum \|x_i\|^p \leq C^p \sup_{x \in B} \sum \|x^* x_i\|^p$$

$\mathcal{M} C = \Pi_p(u)$

$X = Y = H$ and $H$ is $p$-summing, $K \subseteq B$ is $p$-summing

$p = 2$ and $\|x\| = \|x\|_2$

**Definition (Riesz's)** Let $u : X \to Y$ be $p$-summing. Set $\mathcal{M}$

\[ \| x \|_p = \sup_{x \in B} \| x^* x \| \]

For $x^* \in X^*$ and $y \in Y$, $|x^* y| \leq \| x \|_p \| y \|$. So that

$\mathcal{M} \mathcal{M}^* \subseteq \Pi_p(u)$

For $x^* \neq 0$ and $y \neq 0$, $\| x^* y \| \leq \| x \|_p \| y \|$. So that

$\mathcal{M} \mathcal{M} \subseteq \Pi_p(u)$
Example. K compact Haar measure space

\[ X \subseteq (\mathbb{K})_{\| \cdot \|}\]

\[ \eta : X \rightarrow Y \text{ } p\text{-summery}\]

\[ K = \{ \xi_k : k \in K \} \quad \xi_k (x) = f(x_k) \]

Weak compact convex set in \( L_p \)

Pietsch: it is \( p\text{-summing}\) from \( X \) to \( Y \)

There exists a regular Borel probability \( \nu \) on \( K \) such that for every \( x \in X \)

\[ \| h(x) \|_{L^p} \leq \| x \| \]

\[ \| x \|_{L^p} \leq \| x \|_{L^p} \]
\[ \sum_{i} \|u_i\|_L^p \leq n \|g\|_{L^p} \sup_{g} \sum_{i} \|u(x_i)\|^p \]

\[ \text{best constant} \]

\[ Q(x) = \frac{1}{n} \sum_{i} \|x(x_i)\|^p - 1 \]

\[ Q(x) > 0 \text{ somewhere} \]

\[ \mathcal{H} = \{ f \in C(B, \mathbb{R}), \forall x \in B, f(x) < 0 \text{ all } x \} \]

\[ \phi \text{ is convex} \]

\[ \forall \phi \text{ convex on } \mathcal{H} \exists \text{ unique } \mathcal{I} \]

\[ \forall x \in C(B, \mathbb{R}), \exists \mu \geq 0 \text{ all } \mu, \mu(1) > 0 \forall x \]
Compact group

$X \subseteq C(G)$, $X$ is closed, translation invariant

$x \in X$, $g \in G$, $x \cdot (t) = x(\cdot t)$, $x \cdot t \in X$

$\mu \cdot x \rightarrow Y$ $p$-summing

$(X) \cap \mu (x) = ||u(x)|| ||x \cdot t|| + x \cdot t \in G$

Then $\mu$ has a Pietsch measure $\mu^p$

$(x) \cap \mu (x) = \mu (x)$ then $\mu$ is a Pietsch measure for $\mu$.

Such measures exist on $C(G)$, so it's natural to work with from $C(G)$ to $C(G)^X$.

$x \in G$ and in $C(G)^X$, each measure $\mu$.

$\mu \cdot x \rightarrow$ Haar measure of $\mu$ and $\mu$ is translation invariant.
A compact group $G$ acts transitively on the compact space $S$. If there is a map $(g,w) \rightarrow g(w)$ continuous from $G \times S$ to $S$, then

1. $(e,w) = e(w) = w$,
2. $(g, g_2)(w) = g(g_2(w))$,
3. Given $w, w' \in S$. There is a $g \in G$ so $g(w) = w'$.

1, 1. $G$ is a group of homeomorphisms of $S$ onto itself.
2. Says $G$ can move from any point in $S$ to any other.
Example: \[ C/H \text{ closed subgroup of } G \]

\[ C/H \] left coset space

Topological \[ G/H \]

\[ \Pi: G \rightarrow G/H \]

\[ U \in G/H \text{ is open} \quad \Leftrightarrow \quad gH \text{ is open} \]

If \( \Pi \) is continuous, \( H \) being closed as \( G/H \) is Hausdorff.

\[ \text{Hausdorff} \quad G/H \text{ in compact Hausdorff} \]

\[ \text{Transitive action} \quad G \times G/H \rightarrow G/H \]

\[ (g, H) \rightarrow gH \]
Theorem (Weil) If $G$ acts transitively

on $\mathcal{S}$, then there is a closed subgroup $H \leq G$ such that $G$'s action on $\mathcal{S}$ is the same as $G$'s action on $G/H$.

Pick $\omega_0 \in \mathcal{S}$, call it "South Pole". The group $G$ is isomorphic to the group of homeomorphisms $\text{Homeo}(G)$ fixing $\omega_0$ and $\mathcal{S}$, with $G$'s action.
$S^{n-1} = \{ x \in \mathbb{R}^n : ||x|| = 1 \}$

compact sphere

$O(n) = \{ g : \mathbb{R}^n \to \mathbb{R}^n, g \text{ is linear, preserves or permutes sets} \}$

compact group

$O(n)$ acts transitively on $S^{n-1}$

$\{ x_1, \ldots, x_n \} \in \text{OnB of } \mathbb{R}^n$
\[
\sum_{m=0}^{\infty} \mathcal{E}(n) := \left\{ (x_1, \ldots, x_m) \in \mathbb{S}^{m-1} \times \{v_1, \ldots, v_m\} : \text{is even} \right\}
\]

\[
0(n) \text{ acts transitively on } \sum_{m=0}^{\infty} \mathcal{E}(n)
\]

\[
(\nu_1, (x_1, \ldots, x_m)) \mapsto (\nu_1 x_1, \ldots, \nu_1 x_m)
\]

\[
\tilde{g}_m(n) = \text{Grassmann manifold}
\]

\[
\tilde{g}_m(n) = \{ F \in \mathbb{R}^m : \text{linear, m-dim} \}
\]

\[
2^n \Rightarrow \tilde{g}_m(n)
\]

\[
\{x_1, \ldots, x_m\} \mapsto \text{dim}(x_1, \ldots, x_m)
\]
$G \xrightarrow{\text{closed}} G/H$

$B$ Borel set in $G/H$

$\mu_{G/H}(B) = \mu_H(\pi^*_H(B))$

$\mu_{G/H}$ is a regular Borel probability on $G/H$.

$\mu_{G/H}$ is $G$-invariant.

$B$ is $G$-invariant.

$B = \{ x \in G : xH \subseteq B \}$
Theorem (Weyl) Let $G$ be a compact group and suppose $S$ is a compact space on which $G$ acts transitively. Suppose $\mu$ is the normalized Haar measure on $G$ and $\mu/\pi$ is a Haar measure on $\pi(G/H)$. Then $\mu/\pi$ is the unique $G$-invariant, regular Borel probability measure on $G/H$.