Compact topological groups

\( \mu \) normalised Haar measure

bi-invariant, inverse invariant

Unitary representations of \( G \):

\[ x \rightarrow U_x \in \mathcal{L}(H) \]

acting on some Hilbert space \( H \)

\( \forall u, v \in H \), \( \langle U_x u, v \rangle = \langle u, U_x^* v \rangle \), continuous

\( U_1 = U_x u, \quad U_1^* = U_x^* v, \quad U_x = 1 \)

Peter-Weyl Theorem:

If \( G \) is a compact Lie group, \( \mathcal{H} \) is a complete system of irreducible unitary representations on Euclidean spaces.
$x \rightarrow V_x$ representing $G$ on $H$

Irreducible means that if $K \leq H$ and $\forall x K \leq K \forall x$

Then $K = \text{Har}_G K = \{0\}$.

**Theorem (Nach bin)**: If $G$ is a compact group and

$x \rightarrow V_x$ is an irreducible unitary representation of $G$ on the Hilbert space $H$, then

$\dim H < \infty$. 
\[ h \in H \quad x \mapsto \langle x, v \rangle \text{ is a bounded sesquilinear form.} \]

F. Riesz : \( \exists B_h : H \to H \) so that

\[
\langle B_h u, v \rangle = \langle u, v \rangle,
\]

\[
\langle B_h u, v \rangle = \int \langle x, h, u \rangle \langle x, h, v \rangle \, d\mu(x).
\]

\[ B_h v = V_h B_h \]
Schur's lemma

(I) \( x \to U_x, \ x \to V_x \) are irreducible representations of \( G \) and \( F \), respectively.

Imagine \( T \) is an operator.

So \( x \to UV_x = VU_x \) for all \( x \).

Then \( T = 0 \) or \( T \) is invertible.

If \( T = 0 \) it \( x \to U \) is an irreducible representation.

So \( x \to UV_x = VU_x \).

Hence \( T = x \cdot \text{id} \).
\[<B, u, v> = \int_H \phi(x) (u(x) v(x)) \, dx\]
Schur: $B_n = \alpha \text{id}$

\[
\langle B_n \eta, \nu \rangle = \alpha \langle \eta, \nu \rangle
\]

\[
\alpha \leq \alpha \langle \eta \rangle
\]

\[
\int \left| \langle \nu, \eta \rangle \right|^2 = \alpha \langle \eta \rangle \| \eta \|^2
\]

\[
\alpha \langle \eta \rangle \| \eta \|^2 = \int \left| \langle \nu, \eta \rangle \right|^2 = \int \left| \langle \nu, \eta \rangle \right|^2
\]

\[
f(\langle \nu \rangle) = f(\langle \nu \rangle) \quad \text{and} \quad \int f = \int f
\]

\[
\frac{\alpha \langle \eta \rangle}{\| \eta \|^2} = C > 0
\]
\{ q_1, q_2, \ldots, q_n, \ldots \} is an ON system in H

Let \( m \in X \)

\( \{ x_1 \phi_j, x_2 \phi_j, \ldots, x_n \phi_j, \ldots \} \) is an ON system too

\[ \int \langle x_1 \phi_j, \phi_j \rangle \, d\mu(x) = 0, \quad j + k \]

\[ m \leq \sum_{j=1}^{\infty} \int \langle x \phi_j, \phi_j \rangle \, d\mu(x) \]

\[ m \leq \int \exists \quad \mu(x) \leq \| \phi_j \|^2 = 1 \]
Uniformly compact? G locally compact.

Left Haar measure on right Haar measure.

Abelian.

Compact.

Baire: if G is metrizable and has a bi-invariant metric it generates its topology.

$GL_n(R)$ is uniformly compact but its left, right uniform structures are different.
\[ \int x(x) \overline{f(x)} \, dm(x) = 0. \]

\[ x \mapsto u_x \quad \text{unit mass}\]
\[ x \mapsto v_x \quad \text{over } F \text{ dim } F = m \quad \text{on } \{ s_1, \ldots, s_m \} \]

\[ T: E \to F \text{ linear transformation} \]
\[ x \in G \]
\[ v_x T u_x^{-1} = A_x \quad A = \int v_x A_x v_x^{-1} \, dm(x). \]

\[ v_x A u_x^{-1} = A \quad \text{for all } x \]
\[ v_x A u_x = A = 0 \]
\[ 0 = A = \int v_x \overline{f(x)} \, dm(x) \]

\[ Te_i = \chi_i \]
\[ \varphi \equiv 0 \text{ R.T.c.} \]
On the equality

\[ x \mapsto U_x, \quad x \mapsto V_x \text{ are irreducible} \]

Unitary representations are finite-dimensional Hilbert spaces \( E, F \) resp. Suppose these are not equivalent.

\[ U_x = (u_{ij}(x)) \quad V_x = (v_{kl}(x)) \]

\[ V_x^{-1} = V_x^* = (v_{kl}(x)) \]

character of \( U_x \): \( \chi(U_x) \)

\[ \chi(U_x) = \tau_1(U_x) \]

character of \( V_x \): \( \psi(V_x) \)

\[ \psi(V_x) = \tau_1(V_x) \]
\[ \int u_{ij}(x) \sqrt{\nu_h^*(x)} \, d\mu_\lambda(x) = 0 \]

\[ u_x = (u_{ij}(x)) \quad \nu_x = (\nu_{ij}(x)) \]

\[ t_x(u_x) = X(x) \]

\[ \int_X \overline{\nu(x)} \, d\mu_\lambda(x) = 1 \]

\[ T : E \to \overline{E} \]

\[ u_x^T u_x^{-1} = A_x \]

\[ u_x A = A u_x \]

\[ A = \varphi \mathcal{H}_E \]
\[ \chi_{id_{E}} = \sum_{x} U_{x} T U_{x}^{-1} \chi(x) \]

\[ t_{2}(\alpha_{w}, E) = t_{2} \sum_{x} U_{x} T U_{x}^{-1} \chi_{w}(x) \]

\[ \alpha \cdot \dim E = \int_{E} t_{2} U_{x} T U_{x}^{-1} \, d\chi(x) \]

\[ = \int_{E} U_{x} U_{x}^{\ast} \, d\alpha(1) \]

\[ = \int_{E} U_{x}^{\ast} \, d\alpha(1) \]

\[ t_{2}(1) = \alpha \]

\[ \dim \alpha_{w} = t_{2}(1) \]

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\[ \propto \]
Gelfand, Baker, Naimark, Segal

If $G$ is any locally compact group, then $G$ has a complete system of irreducible unitary representations.