CHAPTER 1

Von Neumann's Proof of the Existence and Uniqueness of an Invariant Measure on a Compact Metric group

In this chapter, we'll show how to ascribe to each $f \in C(G)$, a mean $M(f)$, which is at one and the same time, linear in f, non-negative when f is, and is a true average with the values at f and any right translate of f, identical.

Let G be a compact metrizable topological group. Denote by $\mathcal{F}(G)$ the collection of non-empty finite subsets of G and by $C(G)$ the Banach space of all continuous real-valued functions defined on G, equipped as usual with the supremum norm.

Throughout this section, if $F_1, F_2 \in \mathcal{F}(G)$ then by $F_1 \cdot F_2$, we mean all words $a \cdot b$, where $a \in F_1$ and $b \in F_2$; in particular, if $a_1 \cdot b_1 = a_2 \cdot b_2$ but $a_1 \neq a_2$ then we distinguish $a_1 \cdot b_1$ and $a_2 \cdot b_2$.

LEMMA 1.1. (i) *If $f \in C(G)$ then $\min f, \max f,$ and $Osc f = \max f - \min f$ all exist.*
(ii) *If $f \in C(G)$ and $F \in \mathcal{F}(G)$ then*

$$Osc RAve_F f \leq Osc f.$$

In fact,

$$\min f \leq \min RAve_F f \leq \max RAve_F f \leq \max f.$$

(iii) *If $f \in C(G)$ and $F_1, F_2 \in \mathcal{F}(G)$ then*

$$RAve_{F_1} RAve_{F_2} f = RAve_{F_1 \cdot F_2} f.$$

PROOF. To see (ii), let $F \in \mathcal{F}(G)$ and $f \in C(G)$. Define

$$\mathrm{RAve}_{\mathbf{F}}\mathbf{f}(\mathbf{x}) := \frac{\mathbf{1}}{|\mathbf{F}|}\sum_{\mathbf{a}\in\mathbf{F}} \mathbf{f}(\mathbf{xa}), \quad \mathbf{x} \in \mathbf{G}.$$

Naturally $\mathrm{RAve}_F f \in C(G)$.

To see (iii), if $x \in G$ then

$$\begin{aligned}
\mathrm{RAve}_{F_1}\mathrm{RAve}_{F_2}f(x) &= \mathrm{RAve}_{F_1}\frac{1}{|F_2|}\sum_{b\in F_2} f(xb) \\
&= \frac{1}{|F_1|}\sum_{a\in F_1}\frac{S(xa)}{|F_2|} \quad \left(\text{letting } \; S(x) = \sum_{b\in F_2} f(xb)\right) \\
&= \frac{1}{|F_1|\cdot|F_2|}\sum_{a\in F_1} S(xa) \\
&= \frac{1}{|F_1\cdot F_2|}\sum_{a\in F_1}\sum_{a\in F_2} f(xab) \\
&= \frac{1}{|F_1\cdot F_2|}\sum_{c\in F_1\cdot F_2} f(xc) \\
&= \mathrm{RAve}_{F_1\cdot F_2}f(x). \quad \square
\end{aligned}$$

LEMMA 1.2. *If $f \in C(G)$ is not constant then there is an $F \in \mathcal{F}(G)$ such that*

$$OscRAve_F f < Oscf.$$

PROOF. After all, f's not being constant ensures that there is an α such that $\min f < \alpha < \max f$. Set

$$U = [f < \alpha] = \{x \in G : f(x) < \alpha\}.$$

Since $\min f < \alpha$, U is a non-empty open set in G and $G = \bigcup_{a\in G} Ua^{-1}$. (If $x \in G$ then for any $y \in U$, $x = y(y^{-1}x) \in U(y^{-1}x) \subseteq \bigcup_{a\in G} Ua^{-1}$.)

Now U is open (since $f \in C(G)$), and $U \neq \emptyset$ so Ua^{-1} is also a non-empty open set for each $a \in G$. Therefore the Ua^{-1}'s cover the compact G. There is $F \in \mathcal{F}(G)$ such that

$$G = \bigcup_{a\in F} Ua^{-1}.$$

Therefore for any $x \in G$ there exists $a_x \in F$ such that $x \in Ua_x^{-1}$. i.e., for any $x \in G$ there exists $a_x \in F$ such that $f(xa_x) < \alpha$. Thus

$$\begin{aligned}
\mathrm{RAve}_F f(x) &= \frac{1}{|F|}\sum_{a\in F} f(xa) \\
&= \frac{1}{|F|}\left(\sum_{a\in F, a\neq a_x} f(xa) + f(xa_x)\right) \\
&< \frac{1}{|F|}\sum_{a\in F, a\neq a_x} f(xa) + \alpha \\
&\le \frac{(|F|-1)\max f + \alpha}{|F|} \\
&< \frac{(|F|-1)\max f + \max f}{|F|} \\
&= \max f.
\end{aligned}$$

Therefore

$$\text{Osc}\text{RAve}_F f \leq \text{Osc} f.$$

□

LEMMA 1.3. *Let $f \in C(G)$ and define $\mathcal{K} = \{RAve_F f : F \in \mathcal{F}(G)\}$. Then $\mathcal{K}$ is uniformly bounded, equicontinuous family in $C(G)$.*

PROOF. The key to this precious fact is that f is of course uniformly continuous. So given an $\epsilon > 0$ there is an open set V in G containing G's identity such that if $xy^{-1} \in V$ then $|f(x) - f(y)| \leq \epsilon$. Notice that if $a \in G$ and $xy^{-1} \in V$ then $(xa)(ya)^{-1} = xaa^{-1}y^{-1} = xy^{-1} \in V$. So once $xy^{-1} \in V$,

$$|f(xa) - f(ya)| \leq \epsilon$$

for all $a \in G$. But now if $F \in \mathcal{F}(G)$ then whenever $xy^{-1} \in V$ we have

$$\begin{aligned}
|\text{RAve}_F f(x) - \text{RAve}_F f(y)| &= \frac{1}{|F|}\left|\sum_{a\in F} f(xa) - \sum_{a \in F} f(ya)\right| \\
&\leq \frac{1}{|F|}\sum_{a\in F}|f(xa) - f(ya)| \\
&\leq \frac{1}{|F|}|F|\epsilon = \epsilon.
\end{aligned}$$

Note that $\mathcal{K}$ is uniformly bounded since

$$\begin{aligned}
|\text{RAve}_F f(x)| &= \frac{1}{|F|}\left|\sum_{a\in F} f(xa)\right| \\
&\leq \frac{1}{|F|}\sum_{a\in F}|f(xa)| \\
&\leq \frac{1}{|F|}|F| \cdot ||f|| = ||f||_\infty. \quad \square
\end{aligned}$$

We see that Lemma 1.3 takes on added significance if we but recall the classical theory of Arzela and Ascoli to the effect that $\mathcal{K} \subseteq C(G)$ is relatively norm compact if and only if $\mathcal{K}$ is uniformly bounded and equicontinuous.

With Lemmas 1.2 and 1.3 in hand, the plan of attack is clear. We want an averaging technique which will give a true average, assigning values in a uniformly distributed manner, If the function f is constant then we will plainly want to assign that value of constancy to f. With the aforementioned lemmas in hand, we handle non-constant functions thusly; if f is not constant, then we can find $F_1 \in \mathcal{F}(G)$ so that

$$\text{Osc}\text{RAve}_{F_1} < \text{Osc} f;$$

If $\text{RAve}_{F_1}(f)$ is constant then it's value of constancy is the natural value to ascribe to f. If $\text{RAve}_{F_1}(f)$ is not constant, then we appeal to Lemma 1.3 again to find $F_2 \in \mathcal{F}(G)$ so that

$$\text{RAve}_{F_2}\text{RAve}_{F_1} \subset \text{Osc}\text{RAve}_{F_1}(f).$$

Continuing in this vain, we see that in the worst case we can find a sequence $(F_n) \subseteq \mathcal{F}(G)$ so that for each n

$$\text{Osc}\text{RAve}_{F_{n+1}}\text{RAve}_{F_n}(f) \subset \text{Osc}\text{RAve}_{F_n}(f).$$

Appealing to Messrs. Arzela and Ascoli, we can pass to a sequence $(F_n^{'}) \subset \mathcal{F}(G)$ so $(\mathrm{RAve}_{F_n'}(f))$ is uniformly convergent.

The point is that because our averages were taken with respect to right translates, in the long run, judicious choices of the F_n's ought to produce an average that is right invariant. Remarkably enough the wisdom needed has already been provided by Von Neumann.

Lemma 1.4. *Let* $f \in C(G)$ *and* $\mathcal{K} = \{RAve_F f : F \in \mathcal{F}(G)\}$. *Then*

$$\inf_{g \in \mathcal{K}} Osc g = 0.$$

Proof. Let

$$s = \inf_{g \in \mathcal{K}} \mathrm{Osc} g = \inf\{\mathrm{Osc}\mathrm{RAve}_F f : F \in \mathcal{F}(G)\}.$$

Therefore there exists (F_n) in $\mathcal{F}(G)$ such that $(\mathrm{RAve}_{F_n}) \searrow s$. Thanks to Arzela and Ascoli we can, by passing to subsequences if necessary, assume that

$$\mathrm{RAve}_{F_n} f \to g \in C(G),$$

uniformly. It's plain that on assuming the uniform convergence of (RAve_{F_n}) that

$$\min \mathrm{RAve}_{F_n} f \to \min g \text{ and } \max \mathrm{RAve}_{F_n} f \to \max g,$$

and so

$$\mathrm{Osc}\mathrm{RAve}_{F_n} f \to \mathrm{Osc} g.$$

Thus $\mathrm{Osc} g = s$. Here's the point: g is constant! Indeed if g were not constant there would be an $F_0 \in \mathcal{F}(G)$ such that

$$s_0 = \mathrm{Osc}\mathrm{RAve}_{F_0} g < \mathrm{Osc} g = s,$$

thanks to lemma 1.2. Since $(\mathrm{RAve}_{F_n} f)$ is uniformly convergent, there exists N such that

$$||\mathrm{RAve}_{F_N} f - g||_\infty < \frac{s - s_0}{3}.$$

i.e., for any $x \in G$,

$$|\mathrm{RAve}_{F_N} f(x) - g(x)| \leq \frac{s - s_0}{3}.$$

But this is quickly seen to mean

$$|\mathrm{RAve}_{F_0}\mathrm{RAve}_{F_N} f(x) - \mathrm{RAve}_{F_0} g(x)| \leq \frac{s - s_0}{3},$$

for all $x \in G$ as well. It follows that for all $x \in G$

$$|\mathrm{Osc}\mathrm{RAve}_{F_0}\mathrm{RAve}_{F_N} f - \mathrm{Osc}\mathrm{RAve}_{F_0} g| < 2\left(\frac{s - s_0}{3}\right).$$

i.e., for all $x \in G$,

$$|\mathrm{Osc}\mathrm{RAve}_{F_0}\mathrm{RAve}_{F_N} f(x) - s_0| < 2\left(\frac{s - s_0}{3}\right).$$

But this in turn means that

$$\mathrm{Osc}\mathrm{RAve}_{F_0}\mathrm{RAve}_{F_N} f(x) < s_0 + 2\left(\frac{s - s_0}{3}\right) = \frac{2}{3}s + \frac{1}{3}s_0 < s.$$

But

$$\mathrm{Osc}\mathrm{RAve}_{F_0}\mathrm{RAve}_{F_N} f = \mathrm{Osc}\mathrm{RAve}_{F_0 F_N} f,$$

and

$$s = \inf_{F \in \mathcal{F}(G)} \mathrm{OscRAve}_F f.$$

This should elicit an 'OOPS' because

$$\mathrm{RAve}_{F_0}\mathrm{RAve}_{F_N} f = \mathrm{RAve}_{F_0 \cdot F_N} f \in \mathcal{K}.$$

Therefore g is constant and $s = 0$. i.e.,

$$\inf_{g \in \mathcal{K}} \mathrm{Osc} g = 0.$$

□

We say the real number p is a **right mean** of f if for each $\epsilon > 0$ there is an $F \in \mathcal{F}(G)$ such that

$$|\mathrm{RAve}_F f(x) - p| < \epsilon$$

for all $x \in G$. i.e.,

$$||\mathrm{RAve}_F f - p||_\infty < \epsilon.$$

THEOREM 1.5. *Every $f \in C(G)$ has a right mean.*

PROOF. By the techniques used in Lemma 1.4, there is a constant function h (say $h(x) \equiv p$) and a sequence $(F_n) \subseteq \mathcal{F}(G)$ such that

$$\lim_n ||\mathrm{RAve}_{F_n} f - h||_\infty = 0.$$

i.e.,

$$||\mathrm{RAve}_{F_n} f - p||_\infty \to 0,$$

as $n \to \infty$. Plainly p is a right mean of f. □

It's plain that each $f \in C(G)$ has a **left mean** as well, that is, there is a $q \in \mathbb{R}$ so that if $\epsilon > 0$ is given there exists an $F \in \mathcal{F}(G)$ so that

$$\left| \frac{1}{|F|} \sum_{a \in F} f(ax) - q \right| < \epsilon$$

for all $x \in G$. For obvious reasons, we define

$$\mathrm{LAve}_F f(x) = \frac{1}{|F|} \sum_{a \in F} f(ax).$$

THEOREM 1.6. *Let $f \in C(G)$. Let p be a right mean of f and q be a left mean of f. Then $p = q$.*

PROOF. Let $\epsilon > 0$. Find $A, B \in \mathcal{F}(G)$ so that

$$||\mathrm{RAve}_A f - p||_\infty \le \frac{\epsilon}{2}, \quad ||\mathrm{LAve}_B f - q||_\infty \le \frac{\epsilon}{2}.$$

Now

$$\begin{aligned}
\mathrm{RAve}_A\mathrm{RAve}_B f(x) &= \mathrm{RAve}_A \frac{1}{|B|}\sum_{b\in B} f(bx) \\
&= \frac{1}{|A|}\frac{1}{|B|}\sum_{a\in A} S(xa) \ \ (\text{where } \ S(x)=\sum_{b\in B} f(bx)) \\
&= \frac{1}{|A|}\frac{1}{|B|}\sum_{a\in A}\sum_{b\in B} f(bxa) \\
&= \frac{1}{|B|}\frac{1}{|A|}\sum_{b\in B}\sum_{a\in A} f(bxa) \\
&= \frac{1}{|B|}\sum_{b\in B}\frac{1}{|A|}\sum_{a\in A} f(bxa) \\
&= \frac{1}{|B|}\sum_{b\in B}\mathrm{RAve}_A f(bx) \\
&= \mathrm{LAve}_B\mathrm{RAve}_A f.
\end{aligned}$$

Further,

$$\mathrm{RAve}_A(\mathrm{LAve}_B f - q) = \mathrm{RAve}_A\mathrm{LAve}_B f - q$$

and

$$\mathrm{LAve}_B(\mathrm{RAve}_A f - p) = \mathrm{LAve}_B\mathrm{RAve}_A f - p.$$

So for any $x \in G$,

$$\begin{aligned}
|p-q| &= |p-\mathrm{RAve}_A\mathrm{LAve}_B f(x) + \mathrm{RAve}_A\mathrm{LAve}_B f(x) - q| \\
&\le |p-\mathrm{RAve}_A\mathrm{LAve}_B f(x)| + |\mathrm{RAve}_A\mathrm{LAve}_B f(x) - q| \\
&= |p-\mathrm{LAve}_B\mathrm{RAve}_A f(x)| + |\mathrm{RAve}_A\mathrm{LAve}_B f(x) - q| \\
&= |\mathrm{LAve}_B(p-\mathrm{RAve}_A f(x))| + |\mathrm{RAve}_A(\mathrm{LAve}_B f(x) - q)| \\
&\le |p-\mathrm{RAve}_A f(x)| + |\mathrm{LAve}_B f(x) - q| \ \ (\text{since } |\mathrm{LAve}_B f| \le |f| \ \text{ and } \mathrm{RAve}_B f| \le |f|) \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

and $p = q$. Go figure.

Corollary 1.7. *For any $f \in C(G)$ there is a unique number $M(f)$ that is both a right and left mean.*

Theorem 1.8. *The functional M on $C(G)$satisfies the following*

(i) *M is linear.*
(ii) *$Mf \ge 0$ if $f \ge 0$.*
(iii) *$M(1) = 1$.*
(iv) *$M({}_af) = M(f) = M(f_a)$ for each $a \in G$, where ${}_af(x) = f(ax)$ and $f_a(x) = f(xa)$.*
(v) *$M(f) > 0$ if $f \ge 0$ but $f \ne 0$.*
(vi) *$M(\hat{f}) = M(f)$ where $\hat{f}(x) = f(x^{-1})$ for each $x \in G$.*

Proof. We start by showing

$$M(\mathrm{RAve}_F f) = M(f) \tag{0.1}$$

August 26, 2010

for each $f \in C(G)$ and each $F \in \mathcal{F}(G)$. Suppose that $M(f) = p$. If $\epsilon > 0$ is given to us then we can find $F_0 \in \mathcal{F}(G)$ such that

$$||\text{LAve}F_0 f - p||_\infty \leq \epsilon.$$

i.e.,

$$\left| \frac{1}{|F_0|} \sum_{b \in F_0} f(bx) - p \right| \leq \epsilon$$

for all $x \in G$. It follows that for any $x \in G$ and $a \in F$,

$$|\text{RAve}_F \text{LAve}_{F_0} f(x) - p| \leq \epsilon.$$

Since

$$\text{RAve}_F \text{LAve}_{F_0} f = \text{LAve}_{F_0} \text{RAve}_F f,$$

p is a left mean of $\text{RAve}_F f$. Hence by our previous result,

$$M(\text{RAve}_F f) = p,$$

and

$$M(\text{RAve}_F f) = M(f).$$

To see that M is linear, let $M(f) = p$ and $M(h) = q$. Pick $H_0 \in \mathcal{F}(F)$ so that

$$||\text{RAve}_{H_0} h - q||_\infty \leq \epsilon.$$

i.e., for all $x \in G$,

$$\left| \frac{1}{|H_0|} \sum_{b \in H_0} h(xb) - q \right| \leq \epsilon.$$

i.e., if $E \in \mathcal{F}(G)$ and $x \in G$ then

$$|\text{RAve}_{E \cdot H_0} h(x) - q| = |\text{RAve}_E \text{RAve}_{H_0} h(x) - q| < \epsilon.$$

Now

$$p = M(f) = M(\text{RAve}_F f)$$

for any $F \in \mathcal{F}$. Therefore p is the right mean of $\text{RAve}_{H_0} f$. Hence there exists $F_0 \in \mathcal{F}(G)$ so that

$$||\text{RAve}_{F_0} \text{RAve}_{H_0} f - p|| \leq \epsilon.$$

i.e., for all $x \in G$,

$$|\text{RAve}_{F_0 \cdot H_0} f(x) - p| = |\text{RAve}_{F_0} \text{RAve}_{H_0} f(x) - p| \leq \epsilon.$$

Since we already know that for all $x \in G$ and each $E \in \mathcal{F}(G)$,

$$|\text{RAve}_{E \cdot H_0} h(x) - q| \leq \epsilon,$$

it follows that by taking $E = F_0$ we get for each $x \in G$,

$$|\text{RAve}_{F_0 \cdot H_0} (f + h)(x) - (p + q)| \leq 2\epsilon.$$

Thus

$$M(f + h) = M(f) + M(h).$$

It follows from this and the easily established fact that $M(kf) = kM(f)$ that M is linear, and we have shown (i).

Parts (ii) and (iii) are clear. To see (iv), since

$$\text{RAve}_F f(xa) = \text{RAve}_{a \cdot F} f(x), \tag{0.2}$$

$$\begin{aligned} M(f_a) &= M(\mathrm{RAve}_F f_a(x)) \ \text{(by (0.1))} \\ &= M(\mathrm{RAve}_F f(xa)) \\ &= M(\mathrm{RAve}_{a\cdot F} f(x)) \ \text{(by (0.2))} \\ &= M(f) \ \text{(by (0.1))}. \end{aligned}$$

Similarly,

$$\mathrm{LAve}_F f(ax) = \mathrm{LAve}_{F\cdot a} f(x), \tag{0.3}$$

and so

$$\begin{aligned} M({}_a f) &= M(\mathrm{LAve}_F({}_a f(x))) \ \text{(by (0.1) actually it's equivalent with left averages)} \\ &= M(\mathrm{LAve}_F f(ax)) \\ &= M(\mathrm{LAve}_{F\cdot a} f(x)) \ \text{(by (0.3))} \\ &= M(f) \ \text{(by (0.1) actually it's equivalent with left averages)}, \end{aligned}$$

and thus

$$M({}_a f) = M(f) = M(f_a).$$

For (v), suppose that $f \in C(G), f \geq 0, f \not\equiv 0$. Then there is $\alpha > 0$ such that $U = [f > \alpha]$ is non-empty and open; it's easy to see that $\{U_{a^{-1}} : a \in G\}$ is an open cover of the compact G. (If $x \in G$ then for any $y \in G$, $x = y(y^{-1}x) \in U(y^{-1}x) \subseteq \bigcup_{a\in G} Ua^{-1}$.) It follows that for some $a_1, \ldots, a_m \in G$

$$G = Ua_1^{-1} \bigcup Ua_2^{-1} \bigcup \ldots \bigcup Ua_m^{-1}.$$

Let's check to see how this plays out.

If $x \in G$ then $x \in Ua_k^{-1}$ for some $1 \leq k \leq m$. Hence, $xa_k \in U$ and thus $f(xa_k) > \alpha$. It follows that

$$\mathrm{RAve}_{\{a_1,\ldots,a_m\}} f(x) = \frac{1}{m}\sum_{i=1}^{m} f(xa_i) > \frac{\alpha}{m},$$

for all $x \in G$. Therefore

$$0 < \frac{\alpha}{m} \leq M(\mathrm{RAve}_{\{a_1,\ldots a_m\}} f) = M(f).$$

Almost done; we have but to show that $M(f)$ and $M(\hat{f})$ agree. To establish this, define

$$N(f) = M(f \circ \mathrm{inv}),$$

where $\mathrm{inv} : G \to G$ is given by $\mathrm{inv}(x) = x^{-1}$. N is a linear functional on $C(G)$, $N(f) \geq 0$ if $f \geq 0$, and $N(1) = 1$. Moreover

$$\begin{aligned} N({}_a f) &= M(f_a \circ \mathrm{inv}) \\ &= M({}_{a^{-1}}\hat{f}) \ \text{(since } f_a \circ \mathrm{inv}(x) = f_a(x^{-1}) = \hat{f}(a^{-1}x) =_{a^{-1}} f(x)) \\ &= M(\hat{f}) \ \text{(by (iv))} \\ &= N(f). \end{aligned}$$

But by Corollary 1.7, there is only one invariant mean on $C(G)$ so $N(f) = M(f)$. □