CHAPTER 1

Metric Invariance and Haar Measure

Suppose G is a locally compact metrizable group. Then G admits a metric ρ which is left invariant and generates G's topology; G also admits a left invariant regular Borel measure (a left Haar measure). Is there any connection between these left invariant objects? In this chapter we show the answer is an emphatic "yes."

The highlight of the chapter is the wedding of Haar measure and Hausdorff measure; presiding at the ceremony is C. Bandt. The result is a beautiful offspring: *fractional Hausdorff measure.* A consequence is that if λ is a left Haar measure on G and ρ is a left invariant metric generating G's topology then $\rho-$isometric subsets of G have the same $\lambda-$outer measure.

Now the assumption that G have a left invariant metric that generates its group topology is, of course, not an assumption at all. The Birkhoff-Kakutani Theorem assures us that G's metrizability is enough to ensure such a metric exists. However since we are looking for relationships between left invariant measures on G and left invariant metrics on G, it is instructive (and fun!) to show how to generate a left invariant metric that generates G's topology directly from its left Haar measure. This proof is due to R.A. Struble and leads off the chapter.

THEOREM 1.1 (R.A. Struble). *Let G be a locally compact group with left Haar measure λ. Let (V_n) be a decreasing sequence of open sets that form a neighborhood basis of the identity e in G where $\overline{V}_n$ compact for each n. Then*

$$\rho(x,y) = \sup_n \lambda(xV_n \triangle yV_n)$$

defines a left invariant metric on G which is compatible with the topology of G.

PROOF. It's clear that $\rho(x,y)$ is well-defined and that $\rho(x,y) = \rho(y,x)$. Moreover $\rho(x,y) \geq 0$ and $\rho(x,y) < \infty$ regardless of $x, y \in G$ since each V_n is a Borel set with compact closure. Further, $\rho(zx, zy)$ and $\rho(x,y)$ coincide because λ is left invariant.

If $x \neq y$ then there must be an m so that $xV_m \cap yV_m = \emptyset$ since G is Hausdorff; but now

$$\rho(x,y) \geq \lambda(xV_m \triangle yV_m) = 2\lambda(V_m) > 0.$$

On noticing that for any n and any $x, y, z \in G$,

$$xV_n \triangle yV_n \subseteq (xV_n \triangle zV_n) \cup (zV_n \triangle yV_n),$$

we see that for any $x, y, z \in G$,

$$\begin{aligned}\lambda(xV_n \triangle yV_n) &\leq \lambda((xV_n \triangle zV_n) \cup (zV_n \triangle yV_n)) \\ &\leq \lambda(xV_n \triangle zV_n) + \lambda(zV_n \triangle yV_n) \\ &\leq \rho(x,z) + \rho(z,y),\end{aligned}$$

and with this

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y).$$

In sum, ρ is a left invariant metric on G.

If G's topology is discrete then $\lambda(\{e\}) > 0$ so $V_m = \{e\}$ for some m; hence if $x \neq y$,

$$\rho(x,y) \geq \lambda(xV_m \triangle yV_m) = \lambda(\{x,y\}) = 2\lambda(\{e\}) > 0,$$

and the topology induced by ρ is discrete.

If G's topology is *not* discrete then $\lambda(V_n) \searrow \lambda(\cap_n V_n) = \lambda(\{e\}) = 0$. If V is any open set containing e then there is an $m \in \mathbb{N}$ so that $V_m V_m^{-1} \subseteq V$.

Claim 1: $x \in V$ *whenever* $\rho(x,e) < \lambda(V_m)$. To see this, let $\rho(x,e) < \lambda(V_m)$. Then

$$\lambda(xV_m \triangle V_m) \leq \rho(x,e) < \lambda(V_m),$$

a positive number. Were $xV_m \cap V_m = \emptyset$ then

$$\lambda(xV_m \triangle V_m) = 2\lambda(V_m) < \lambda(V_m),$$

oops! So $xV_m \cap V_m \neq \emptyset$ and thus there are $v_1, v_2 \in V_m$ so $xv_1 = v_2 \in xV_m \cap V_m$ and

$$x = v_2 v_2^{-1} \in V_m V_m^{-1} \subseteq V.$$

This is so whenever $\rho(x,e) < \lambda(V_m)$, and our claim is justified.

Let's look at all of the points x such that $\rho(x,e) < r$, where $r \in \mathbb{Q}, r > 0$. There must be an $m \in \mathbb{N}$ so that $\lambda(V_n) < \frac{r}{4}$, whenever $n \geq m$. Each of the functions

$$f_k(x) = \lambda(xV_k \triangle V_k)$$

is continuous and satisfies $f_k(e) = \lambda(V_k \triangle V_k) = \lambda(\emptyset) = 0$. But now we know there is an $l \in \mathbb{N}$ so that if $x \in V_l$ then $f_1(x), \ldots, f_{m-1}(x) < r$.

Claim 2: *if* $x \in V_l$ *then* $\rho(x,e) < r$. Why is this so? Well if $x \in V_l$ then by choice of $l \in \mathbb{N}$, we have

$$\lambda(xV_1 \triangle V_1), \ldots, \lambda(xV_{m-1} \triangle V_{m-1}) < r.$$

What about $\lambda(xV_k \triangle V_k)$ for $k \geq m$? In this case,

$$\lambda(xV_k \triangle V_k) \leq 2\lambda(V_k) < 2 \cdot \frac{r}{4} < r.$$

It follows that $\rho(x,e) = \sup_n \lambda(xV_m \triangle V_n) < r$.

Our two claims taken in tandem show that ρ generates G's topology about e. Since ρ is left invariant and since G's topology is too, this is enough to say that ρ generates G's topology everywhere. $\square$

THEOREM 1.2 (C. Bandt). *If ρ is a left invariant metric on the locally compact metrizable group G defining the topology of G, then any two subsets of G that are ρ-isometric have the same left Haar measure.*

To prove Theorem 1.2 we need to develop fractional Hausdorff measure. let A be a fixed compact set with non-empty interior;

We'd like to construct a Hausdorff gauge function h so that the associated Hausdorff measure μ^h on G satisfies

$$0 < \mu^h(A) < \infty.$$

Sadly finding such a gauge function is elusive. Happily Bandt found a way around this: fractional Hausdorff measure. Given a Hausdorff gauge function h we define the **fractional Hausdorff measure** ν^h by

$$\nu^h(E) = \lim_{t\searrow 0} \inf\left\{\sum_j c_j h(\mathrm{diam}(B_j)) : c_j > 0, \chi_E \le \sum_j c_j \chi_{B_j}, \mathrm{diam}(B_j) \le t\right\}.$$

Mimicking the proofs encountered in Hausdorff measures, we see that ν^h is a metric outer measure (ensuring us that Borel sets are ν^h-measurable) and ν^h is left invariant reflecting ρ's left invariance. The issue is to judiciously choose h so that

$$0 < \nu^h(A) < \infty.$$

Once this has been achieved, we see that for any compact set $K \subseteq G, \nu^h(K) < \infty$ (K can be covered by finitely many of A's left translates, each of which has the same ν^h-measure) and for any non-empty open subset U of G, $\nu^h(U) > 0$ (we can cover A by finitely many left translates of U, each having the same ν^h-measure).

Before proceeding we take note that if $0 < s < t$ then for any $E \subseteq G$

$$\inf\left\{\sum_j c_j h(\mathrm{diam}(B_j)) : c_j > 0, \chi_E \le \sum_j c_j \chi_{B_j}, \mathrm{diam}(B_j) \le t\right\} \tag{1.1}$$

is at least as large

$$\inf\left\{\sum_j c_j h(\mathrm{diam}(B_j)) : c_j > 0, \chi_E \le \sum_j c_j \chi_{B_j}, \mathrm{diam}(B_j) \le s\right\};$$

after all, there are at least as many fractional coverings of E when the sets B_j have diameter less than or equal to s as there are with sets B_j having diameter less than or equal to t. So, as function of t, (1.1) ascends as t descends to zero. It follows that $\nu^h(E)$ *is determined by what happens when the fractional coverings involve sets of small diameter.*

The gauge function that works is

$$h(t) = \sup\{\lambda(B) : \mathrm{diam}(B) := \mathrm{diam}(B) \le t\},$$

a function that takes finite values on some open interval $(0, T_0)$. Indeed, G is locally compact and so G has a basis for its open sets consisting of sets with compact closure; it follows that at any point of G, we have a basis of open balls with compact closure all with diameter less than T_0, for some $T_0 > 0$. The left invariance of ρ ensures that the same T_0 works throughout G.

NOTE 1.3. *ν^h is left invariant.*

As a matter of fact, the metric ρ of Theorem 1.1 that generates G's topology is left invariant and so for any subset B of G, the diameter of B and gB are the same, since they're $\rho-$isometric.

By the same token, sets in G that are isometric with respect to ρ are assigned the same ν^h-values.

The real issue with ν^h is to show that it's non-trivial, that is, $0 < \nu^h(A) < \infty$. Once we know this to be so then ν^h is a left Haar measure on G and so is but a multiple of λ. Hence sets in G that are isometric (with respect to ρ) have the same $\lambda-$measure as well and we will have proved Theorem 1.2. We proceed with several lemmas to get that ν^h is non-trivial.

LEMMA 1.4. *For any Borel set $B \subseteq G$,*

$$\lambda(B) \leq \nu^h(B).$$

PROOF. Suppose $\chi_B \leq \sum_j c_j \chi_{B_j}$. Then $\chi_B \leq \sum_j c_j \chi_{\overline{B}_j}$ and so

$$\begin{aligned}
\lambda(B) &= \int_B d\lambda \leq \int \sum c_j \chi_{\overline{B}_j}\, d\lambda \\
&= \sum c_j \lambda(\overline{B}_j) \leq \sum c_j h(\mathrm{diam}(\overline{B}_j)) \\
&= \sum c_j h(\mathrm{diam}(B_j)).
\end{aligned}$$

It follows that

$$\lambda(B) \leq \nu^h(B). \quad \square$$

Of course a particular consequence of this lemma is

COROLLARY 1.5. $0 < \lambda(A) < \nu^h(A)$.

LEMMA 1.6. *Let $E(t)$ be defined by*

$$E(t) = \inf \left\{ \sum_{j=1}^n c_j : n \in \mathbb{N}, \chi_A \leq \sum_{j=1}^n c_j \chi_{B_j}, c_j \geq 0, diam(B_j) \leq t \right\}.$$

If

$$\liminf_t h(t)E(t) < \infty$$

then $\nu^h(A) < \infty$.

PROOF. There is a $c > 0$ and a sequence (t_k), $t_k > 0$ with $t_k \searrow 0$ so that

$$h(t_k)E(t_k) < c$$

for all k. In other words,

$$E(t_k) < \frac{c}{h(t_k)}$$

for all k. For each k, choose a fractional covering of A,

$$\chi_A \leq \sum_{j=1}^{n(k)} c_j^{(k)} \chi_{B_j^{(k)}}, \quad c_j^{(k)} \geq 0$$

with

$$\sum_{j=1}^{n(k)} c_j^{(k)} \leq \frac{c}{h(t_k)}.$$

Of course, the definition of the fractional Hausdorff measure $\nu^h(A)$ ensures that

$$\nu^h(A) \leq \inf\left\{\sum_{j=1}^{n} c_j h(\mathrm{diam}(B_j)) : c_j \geq 0, \chi_A \leq \sum c_j \chi_{B_j}, \mathrm{diam}(B_j) \leq t_k\right\}$$

so

$$\nu^h(A) \leq \sum_{j=1}^{n(k)} c_j^{(k)} h(t_j) \leq c < \infty. \quad \square$$

Lemmas 1.4 and 1.6 show us the way to the end, namely the proof Bandt's theorem, which will follow from the following.

Lemma 1.7 (Principal Lemma). *For each $\epsilon > 0$ there is a $t_0 > 0$ so that if U is an open subset of G with diam$(U) \leq t_0$ then for some $s_1, \ldots, s_n \in G$ and $\alpha_1, \ldots, \alpha_n > 0$ we have*

$$\chi_A \leq \sum_{i=1}^{n} \alpha_i \chi_{s_i \cdot U}$$

and

$$\lambda(A) \leq \sum_{i=1}^{n} \alpha_i \lambda(s_i \cdot U) \leq (1+\epsilon)\lambda(A).$$

This proof depends on Cartan's Approximation Theorem (Theorem **??**) and is rather delicate. We postpone the proof until after seeing what it buys us - the completion of the proof of Bandt's theorem. So principal lemma in hand, let's show how to put Lemmas 1.4 and 1.6 into play.

Corollary 1.8. *The fractional Hausdorff measure is non-trivial; in fact,*

$$0 < \nu^h(A) < \infty.$$

Proof. If $\epsilon > 0$ then we can choose $t > 0$ so that $t < \min\{t_0, \epsilon\}$ and h is continuous at t. We can do this since h is monotone and so is continuous at all but countably many points of $(0, \min\{t_0, \epsilon\})$.

We can find an open set B in G with diam$(B) \leq t$ so that

$$h(t) \leq (1+\epsilon)^2 \lambda(B).$$

How can we do this? Well if we pick $t^{'} < t$ so that $h(t^{'}) \leq h(t)$, and since h is continuous at t,

$$h(t) < (1+\epsilon)h(t^{'}),$$

then we choose C so that diam$(C) \leq t^{'}$ and $h(t^{'}) \leq (1+\epsilon)\lambda(C)$. Then

$$B = \left\{x \in G : \rho(x, C) < \frac{t - t^{'}}{2}\right\}$$

will do. By our Principal Lemma we have a functional covering of A: there is $g_1, \ldots, g_n \in G$ and $\alpha_1, \ldots, \alpha_n > 0$ so that

$$\chi_A \leq \sum_{i=1}^{n} \alpha_i \chi_{g_i \cdot B}$$

with

$$\lambda(B)(\sum_{i=1}^{n} \alpha_i) \leq \sum_{i=1}^{n} \alpha_i \lambda(\alpha_i \cdot B) \leq (1+\epsilon)\lambda(A).$$

It follows that

$$\begin{aligned} h(t)E(t) &= h(t)E(t) \leq h(t) \sum_{i=1}^{n} \alpha_i \\ &\leq (1+\epsilon)^2 \lambda(U) \sum_{i=1}^{n} \alpha_i \\ &\leq (1+\epsilon)^3 \lambda(A). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary,

$$\liminf_{t \searrow 0} h(t)E(t) \leq \lambda(A) < \infty,$$

and so by Lemma 1.6, $\nu^h(A) < \infty$. We already know that $0 < \nu^h(A)$ and so by Note 1.3, ν^h is $\rho-$invariant with $0 < \nu^h(A) < \infty$. □

Proof. (of Principal Lemma) Since λ is regular and A is compact we can find an open set V such that

$$V \subseteq \{x \in G : \rho(x, A) < b\}$$

that contains A, has $\overline{V}$ compact and satisfies

$$\lambda(V) \leq (1+\epsilon)^{1/3}\lambda(A).$$

To see this, for each $a \in A$, let U_a be an open set containing a such that

$$U_a \subseteq \overline{U}_a \subseteq \{x : \rho(x, A) < b\}.$$

Let $a_1, \ldots, a_n \in A$ so that

$$A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}.$$

Then

$$V = U_{a_1} \cap \cdots \cap U_{a_n}$$

is open, contains A, $\overline{V}$ is compact and $\overline{V} \subseteq \{x : \rho(x, A) < b\}$.

Let

$$W = \{x \in U : \rho(x, A) < b/2\}.$$

Let $f : G \rightarrow [0, 1]$ be a continuous function that is one on A and vanishes outside of W. Choose $\alpha > 0$ so that $\alpha[1 + (1+\epsilon)^{1/3}] < (1+\epsilon)^{1/3} - 1$, that is, $\alpha < 1 - (1+\epsilon)^{-1/3}$. Then

$$\frac{1+\alpha}{1-\alpha} < (1+\epsilon)^{1/3}.$$

We appeal to Cartan's Approximation Scheme to get an open set U_0 that contains G's identity e for which if $\phi \in \mathcal{K}^+(G)$ and U_0 contains the support of ϕ then for some $g_1, \ldots, g_n \in \mathrm{supp}(f)$ and some $c_1, \ldots c_n \geq \alpha$ we have

$$\left| f(g) - \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g) \right| \leq \alpha$$

for all $g \in G$. Let $t_0 < \min\{b/2, d(e, U_0^c)\}$ be a positive number. We'll show that this is the t_0 claimed in the Principal Lemma.

Let B be an open set with $\mathrm{diam}(B) \leq t_0$. Notice that if $g \in B$ then $g^{-1}B$ is an open set that contains e and $\mathrm{diam}(g^{-1}B) = \mathrm{diam}(B) \leq t_0$, since d is left invariant. Each point of B is within t_0 of g and so each point of $g^{-1}B$ is within t_0 of e. Can any $x \in g^{-1}B$ also be in U_0^c? If we try to imagine such an x then

$$d(e, x) \leq t_0 < d(e, U_0^c),$$

an impossibility. So (replacing B with $g^{-1}B$ if necessary), we can assume our B in the opening line of this paragraph contains e and is open with $\mathrm{diam}(B) \leq t_0$ and $B \subseteq U_0$.

Now λ is inner regular so we can choose a compact $C \subseteq B$ so $\lambda(C)$ is almost $\lambda(B)$, that is,

$$\lambda(B) \leq (1 + \epsilon)^{1/3} \lambda(C).$$

Suppose $\phi : G \to [0, 1]$ is a continuous function for which

$$\chi_C \leq \phi \leq \chi_B \leq \chi_{U_0}.$$

We know that ϕ is a member of $\mathcal{K}^+(G)$ with $\mathrm{supp}(\phi) \subseteq U_0$ and so Cartan's scheme tells us we can find $g_1, \ldots, g_n \in \mathrm{supp}(f)$ and $c_1, \ldots, c_n \geq 0$ so that for any $g \in G$

$$\left| f(g) - \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g) \right| \leq \alpha;$$

alternatively,

$$f(g) - \alpha \leq \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g) \leq f(g) + \alpha$$

for any $g \in G$. Since $\chi_A \leq f$,

$$\chi_A(g) - \alpha \leq \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g)$$

for any $g \in G$. Let $\phi_i = {}_{g_i}\phi$ and $d_i = \frac{c_i}{1-\alpha}$. Plainly $\phi_i \leq \chi_{g_i B}$ (since $\phi \leq \chi_B$) and

$$\chi_A \leq \sum_{i=1}^{n} d_i \phi_i \leq \sum_{i=1}^{n} d_i \chi_{g_i B}.$$

Now $\sum_{i=1}^{n} d_i \chi_{g_i B}$ is zero in V^c; after all, $g_1, \ldots, g_n$ are in the support of f and

$$d(W, V^c) > b/2 \geq \mathrm{diam}(B)$$

so

$$\sum_{i=1}^{n} d_i \phi_i \leq (1 + \epsilon)^{1/3} \chi_V.$$

It follows that

$$\begin{aligned}\sum_{i=1}^n d_i \int \phi \, d\lambda &= \sum_{i=1}^n d_i \int \phi_i \, d\lambda = \int \sum_{i=1}^n d_i \phi_i \, d\lambda \\ &\leq (1+\epsilon)^{1/3} \int \chi_V \, d\lambda = (1+\epsilon)^{1/3} \lambda(V).\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{i=1}^n d_i \lambda(g_i B) &= \lambda(B) \sum_{i=1}^n d_i \leq (1+\epsilon)^{1/3} \lambda(C) \sum_{i=1}^n d_i \\ &\leq (1+\epsilon)^{1/3} \int \phi \, d\lambda \sum_{i=1}^n d_i \leq (1+\epsilon)^{2/3} \lambda(V) \leq (1+\epsilon)\lambda(A)\end{aligned}$$

and that's that. □

Proof. (of Bandt's Theorem - Theorem 1.2) By Corollary 1.8 ν^h is a non-trivial left invariant Haar measure on G, so it is a multiple of λ. Since sets in G which are $\rho-$isometric have the same ν^h values, it follows that these sets have the same left Haar measure. □

1. Notes and Remarks

Whenever G is a locally compact metrizable topological group, G has a base for its topology consisting of open sets with compact closure; the collection of open balls with respect to the metric generates G's topology also forms a base for its topology. When can one find a left-invariant metric generating G's topology all of whose balls have compact closure? Of course, for such a thing to be so, G must be separable: after all, G is the union of the $n-$balls centered at the identity; if each of these have compact closure then it's easy to see that G has a countable dense subset - the union of the countable dense subsets of each $n-$ball will do.

Our next result, also due to R.A. Struble, tells us that this is the whole story. Though the topic of this result of Struble is not germane to the study of Haar measure, it is simply too satisfying a result not to be included.

Here's the theorem of R. Struble that led him to consider Theorem 1.1.

Theorem 1.9. *A locally compact group metrizable topological group has a left invariant metric that generates its topology in which all its open balls have compact closure if and only if G satisfies the second axiom of countability.*

Lemma 1.10. *Let G be a locally compact, second countable (hence metrizable, separable) group. Then there exists a family $\{U_r : r > 0\}$ such that*

(i) for each r, each U_r is open and $\overline{U}_r$ is compact,
(ii) $U_r = U_r^{-1}$
(iii) $U_r U_s \subseteq U_{r+s}$ (so if $r < s$ then $U_r \subseteq U_r U_{r-s} \subseteq U_s$),
(iv) $\{U_r : r > 0\}$ is a base for the open sets about e,
(v) $\cup_{r>0} U_r = G$.

Once Lemma 1.10 is established, we're ready for business. Indeed, let $\{U_r : r > 0\}$ be the family of open sets about e generated from Lemma 1.10. For $x, y \in G$, set

$$d(x,y) = \inf\{r : y^{-1}x \in U_r\}.$$

- Since $G = \cup_{r>0} U_r$, for an pair $x, y \in G$, we have $y^{-1}x \in U_r$ for some $r > 0$. It follows that $d(x,y) \geq 0$.
- $e \in U_r$ for each $r > 0$ so $d(x,x) = 0$.
- If $y^{-1}x \neq e$ then there is an $r_0 > 0$ so that $y^{-1}x \notin U_{r_0}$ (Part (iv) of Lemma 1.10 tells us this). But whenever $0 < r_0 < r$ we have (by Part (iii) of Lemma 1.10)

$$U_{r_0} \subseteq U_{r_0} U_{r-r_0} \subseteq U_r,$$

  so $d(x,y) \geq r_0 > 0$.
- $U_r = U_r^{-1}$ so $y^{-1}x \in U_r$ precisely when $x^{-1}y \in U_r$; consequently, $d(x,y) = d(y,x)$.
- Suppose $x, y, z \in G$ with $y^{-1}x \in U_r, z^{-1}y \in U_s$. Then

$$z^{-1}x = z^{-1}yy^{-1}x \in U_s U_r \subseteq U_{r+s},$$

  so $d(x,z) \leq r + s$. This is so whenever $y^{-1}x \in U_r$ so $d(x,z) \leq d(x,y) + s$; again this is so whenever $z^{-1}y \in U_s$ so $d(x,z) \leq d(x,y) + d(y,z)$.
- Finally, if $x, y, z \in G$ then

$$d(zx, zy) = \inf\{r : (zy)^{-1}zx \in U_r\} = \inf\{r : y^{-1}x \in U_r\} = d(x,y).$$

To summarize: d is a left invariant metric on G.

Since $d(x,e) < r$ means $x = e^{-1}x \in U_r$, the open d-ball of radius r centered at e is contained in U_r. Also this same d-ball contains $U_{r'}$ for any $0 < r' < r$ since if $x \in U_{r'}$ then

$$e^{-1}x = x \in U_{r'} \subseteq U_{r'} U_{\frac{r-r'}{2}} \subseteq U_{\frac{r+r'}{2}},$$

and so $d(x,e) \leq \frac{r+r'}{2} < r$. Therefore if $0 < r' < r$ then

$$U_{r'} \subseteq \{x : d(x,e) < r\} \subseteq U_r.$$

It follows that the open $d-$balls of radius r about e are cofinal with the collection $\{U_r : r > 0\}$ so the closure of each open $d-$ball is compact and d generates G's topology.

PROOF. (Lemma 1.10) Let ρ be the left invariant metric resulting from Theorem 1.1. We can assume that each of the open balls

$$B_r = \{x \in G : \rho(x,e), r\}$$

has compact closure for $0 < r \leq 2$; after all, there is an r_0 so that for $r < r_0$, $\overline{B}_{r_0}$ is compact by G's locally compact nature so recalibrate ρ to make $r_0 = 2$ if necessary.

For $0 < r < 2$ we let $U_r = B_r$. This assures us clearly of (iv) and since we'll keep these U_r's, (iv) is assumed henceforth. Also (i), (ii), and (iii) hold when $r+s < 2$ by ρ's left invariant metric nature.

G is locally compact and satisfies the second countability axiom so G admits a countable open base

$$\{W_{2^n} : n \in \mathbb{N}\}$$

for its topology, where we can (and do) assume that $\overline{W}_{2^n}$ is compact for each n. We define

$$U_2 = B_2 \cap W_2.$$

It's easy to verify that (i) and (ii) hold for $0 < r < 2$ and if $r + s < 2$ then (iii) holds as well.

We'll now inch our way from from (i), (ii), and (iii), $(r + s \leq 2)$ holding for $0 < r \leq 2$ to $0 \leq r \leq 4$.

First we have to define U_r for $2 < r < 2^2$. Let $0 < r < 2^2$. Set

$$U_r = \bigcup U_{t_1} \cdots U_{t_m}$$

where the union extends over all $t_1, \ldots t_m$ so that each t_i satisfies $0 < t_i \leq 2$ and $t_1 + \cdots + t_m = r$.

If $2 < r < 2^2$ and $t_1 + \cdots + t_m = r$ where each $t_i > 0$ then there must be $k, l \in \mathbb{N}$ so that $1 \leq k < l < m$ and $t_1 + \cdots + t_k \leq 2$, $t_{k+1} + \cdots + t_l \leq 2$, and $t_{l+1} + \cdots + t_m \leq 2$. Why is this so? Well let k be the least j_1 so that $t_1 + \cdots + t_{j_1} \leq 2$, and let l be the least j_2 so that $t_{j_1+1} + \cdots + t_{j_2} \leq 2$. Then $\sum_{j_2+1}^{m} t_j \leq 2$ because otherwise, $t_{j_k+1} + \cdots + t_m > 2$ and $t_1 + \cdots + t_{j_1+1} \geq 2$ too where $j_1 + 1 < j_2 + 1$.

It follows that

$$\begin{aligned} U_{t_1} \cdot U_{t_2} \cdots U_{t_m} &\subseteq (U_{t_1} \cdots U_{t_k})(U_{t_{k+1}} \cdots U_{t_l})(U_{t_{l+1}} \cdots U_{t_m}) \\ &\subseteq U_{t_1 + \cdots + t_k} U_{t_{k+1} + \cdots + t_l} U_{t_{l+1} + \cdots + t_m} \quad \text{by (iv)} \\ &\subseteq U_2 \cdot U_2 \cdot U_2, \end{aligned}$$

so $U_r \subseteq U_2 \cdot U_2 \cdot U_2$ whenever $0 < r < 2^2$. Now $\overline{U}_2$ is compact so $\overline{U_2 \cdot U_2 \cdot U_2} \subseteq \overline{U}_2 \cdot \overline{U}_2 \cdot \overline{U}_2$ is too and $\overline{U}_r$ is compact for $0 < r < 2^2$. Since

$$(U_{t_1} \cdots U_{t_m})^{-1} = U_{t_m}^{-1} \cdots U_{t_1}^{-1} = U_{t_m} \cdots U_{t_1},$$

we see that

$$\begin{aligned} U_r^{-1} &= \left(\bigcup U_1 \cdots U_{t_m} \right)^{-1} \\ &= \bigcup (U_1 \cdots U_{t_m})^{-1} \\ &= \bigcup U_{t_m}^{-1} \cdots U_1^{-1} \\ &= \bigcup U_{t_m} \cdots U_1 = U_r \end{aligned}$$

for $2 < r < 2^2$.

Finally if $r > 0, s > 0$ and $r + s < 2^2$ then on supposing $t_1, \ldots, t_m, \tau_1, \ldots, \tau_j$ are positive and satisfy $t_1 + \cdots + t_m = r, \tau_1 + \cdots + \tau_j = s$ then

$$(U_{t_1} \cdots U_{t_m})(U_{\tau_1} \cdots U_{\tau_j}) = U_{t_1} \cdots U_{t_m} U_{\tau_1} \cdots U_{\tau_j}$$

so

$$U_r \cdot U_s \subseteq U_{r+s}.$$

What if $r, s > 0$ and $r + s = 2^2$? Suppose $r = s = 2$. Then $U_r U_s = U_2 U_2$. If $2 < r$ then $s < 2$. But $r < 2^2$ so

$$U_r \subseteq U_2 U_2 U_2$$

and so

$$U_r U_s \subseteq (U_2 U_2 U_2) U_2.$$

Either way define

$$U_{2^2} = (U_2 U_2 \cdot U_2 U_2) \cup W_{2^2}.$$

Here's what is so for $0 < r < 2^2$:

- Each U_r is open and $\overline{U}_2$ is compact;
- $U_r^{-1} = U_r$;
- if $r, s > 0$ and $r + s \leq 2^2$ then $U_r U_s \subseteq U_{r+s}$.

We still have $\{U_r : 0 \leq r \leq 2^2\}$ as a basis for the topology of G about e and of course, $W_{2^2} \subseteq U_{2^2}$.

We continue from U_{2^2} to U_{2^3} in a similar fashion and inch our way forward in a straightforward modification of the above procedure.

The fact that at each stage we ensure $W_{2^n} \subseteq U_{2^n}$ allows us to conclude that

$$G = \bigcup_{n \in \mathbb{N}} W_{2^n} \subseteq \bigcup_{n \in \mathbb{N}} U_{2^n} \subseteq G$$

which is (v) of Lemma 1.10. □

THEOREM 1.11 (Braconnier). *If G is a locally compact topological group and G admits a bi-invariant metric d that determines its topology then G is unimodular.*

PROOF. Let λ be left Haar measure on G and assume that G is not unimodular. Let U be an open set containing e such that $\lambda(U) < \infty$. Let B be an open ball centered at e (of radius ρ) and so that $\overline{B}$ is compact. Then for any $x \in G$, the set xBx^{-1} is just B, thanks to d's bi-invariance, so

$$xBx^{-1} \subseteq U.$$

Let $x_0 \in G$ satisfy

$$\Delta(x_0^{-1})\lambda(B) > \lambda(U),$$

where Δ is the modular function of G. Then $x_0 B x_0^{-1} \subseteq U$ and

$$\lambda(x_0 B x_0^{-1}) = \Delta(x_0^{-1})\lambda(x_0 B) = \Delta(x_0^{-1})\lambda(B) > \lambda(U).$$

Oops! □

EXAMPLE 1.12. *The general linear group, $\mathcal{GL}(n; \mathbb{R})$, on $n-$space, $n \geq 2$, is unimodular. For each m, let*

$$X_m = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ 0 & m \end{pmatrix}, \quad \textit{and} \quad Y_m = \begin{pmatrix} m & \frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix} \in \mathcal{GL}(n; \mathbb{R}).$$

Then

$$X_m Y_m = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ 0 & m \end{pmatrix} \begin{pmatrix} m & \frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{m^2} \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But

$$Y_m X_m = \begin{pmatrix} m & \frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix} \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ 0 & m \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Therefore no bi-invariant metric can be found so that it generates the topology of $\mathcal{GL}(n; \mathbb{R})$.

THEOREM 1.13. *If G is a locally compact metrizable topological group and ρ is a left invariant metric that generates G's topology then (G, ρ) is a complete metric.*

Indeed if U is an open set with compact closure and if $e \in U$ then there is an open ball B centered at e with $\overline{B}$ both compact and contained in U. Suppose R is the radius of B and let (g_n) be a $\rho-$Cauchy sequence in (G, ρ). Then there is an $N \in \mathbb{N}$ so for $m, n \leq N$,

$$\rho(g_n, g_m) < \frac{R}{3};$$

it soon follows that for $n \geq N$,

$$\rho(g_n, g_N) < \frac{R}{3}.$$

Therefore for $n \geq N$, g_n lies in the compact, closed ball of radius $\frac{R}{3}$ and so (g_n) must converge.

Observation 1.14. *Let U be an open set in the topological group G with $e \in G$ and suppose that K is a compact subset of G. Then there is an open set V in G with $e \in V$ so that*

$$xVx^{-1} \subseteq U$$

for every $x \in K$.

Proof. To see this, let $\mathcal{W}$ denote the collection of all open sets W in G such that $W = W^{-1}$. We claim that for any $y \in G$ there is a $V \in \mathcal{W}$ so that if $x \in Vy$ then

$$xVx^{-1} \subseteq U.$$

In fact, we can pick $V_1 \in \mathcal{W}$ so that $V_1 \cdot V_1 \cdot V_1 \subseteq U$ and we can pick $V_2 \in \mathcal{W}$ so that

$$yV_2y^{-1} \subseteq V_1.$$

(This is thanks to the continuity of $x \to ax \to axa^{-1}$ for any $a \in G$.) Let

$$V = V_1 \cap V_2.$$

Then if $x \in Vy$,

$$xy^{-1} \in V \subseteq V_1,$$

and

$$yx^{-1} = (x^{-1}y)^{-1} \in V^{-1} = V \subseteq V_1,$$

and so

$$xVx^{-1} \subseteq xV_2x^1 = (xy^{-1})\left(yV_2y^{-1})\right)(yx^{-1}) \subseteq V_1 \cdot V_1 \cdot V_1 \subseteq U.$$

So for each $y \in K$ there is a $V_y \in \mathcal{W}$ so that $x \in V_xy$ implies $xV_yx^{-1} \subseteq U$. But

$$K \subseteq \bigcup_{y \in K} V_yY,$$

and each V_yy is open; hence we can find $y_1, \ldots, y_n \in K$ so that

$$K \subseteq (V_{y_1}y_1) \cup \cdots \cup (V_{y_n}y_n).$$

Let

$$V = V_{y_1} \cap \cdots \cap V_{y_n}.$$

Then if $x \in K$, it must be that $x \in V_{y_k}y_k$ for some $k, 1 \leq k \leq n$, and so

$$xVx^{-1} \subseteq xV_{y_k}y_k \subseteq U.$$

□

Theorem 1.15. *A compact metrizable topological group G admits a bi-invariant metric that generates its topology.*

PROOF. Let ρ be a left invariant metric on G that generates G's topology. For $x, y \in G$ define

$$d(x, y) = \sup\{\rho(xz, yz) : z \in G\}.$$

Then d is finite for all $x, y \in G$ and is easily seen to be bi-invariant.

Suppose $\epsilon > 0$. By our observation above, there is a $\delta > 0$ so that

$$z^{-1} \cdot \{x \in G : \rho(x, e) < \delta\} \cdot z \subseteq \{x \in G : \rho(x, e) < \epsilon\},$$

for all z. It follows that $\rho(x, e) < \delta$ ensures that $d(z^{-1}xz, e) = d(xz, z) < \epsilon$ for all z and so $d(x, e) \leq \epsilon$. The open $\rho-$ball of radius δ centered at e is contained in the closed $d-$ball of radius ϵ centered at e.

It is plain that the open $d-$ball of radius ϵ centered at e is contained in the open $\rho-$ball of radius ϵ centered at e. Therefore ρ and d generate the same topology. □

Moreover

- *If G is a topological group of the second category and H is a subgroup of G then $G\backslash H$ is either empty or of the second category in G.*

  Let $y \in G\backslash H$. Then $yH \in G\backslash H$ (distinct cosets are disjoint). Therefore should $G\backslash H$ be of the first category then so is yH and from this we conclude that $G = yH \cup (G\backslash H)$ is of the first category.
- *If G is a topological group of the second category and H is a dense $\mathcal{G}_\delta$ subgroup of G then $H = G$.*

  After all, $H = \cap_n H_n$ where each H_n is a dense open subset of G and so $G\backslash H_n$ is closed and nowhere dense for each n; it follows that $G\backslash H = \cup_n(G\backslash H_n)$ is of the first category, and so by the previous remark, $G\backslash H = \emptyset$.

THEOREM 1.16 (V.Klee). *Let G be a topological group with a bi-invariant metric ρ which generates G's topology. Suppose (G, ρ) admits a complete metric d that generates G's topology. Then G is actually complete under ρ.*

PROOF. Let (G^*, ρ^*) be the completion of (G, ρ) Then (G^*, ρ^*) is a topological group into which (G, ρ) is naturally isomorphically and isometrically embedded as a dense subgroup. But topological complete metric spaces are always $\mathcal{G}_\delta-$sets in any super metric space, thanks to an oldie but goodie of Sierpinski. Hence $G = G^*$ and so G is $\rho-$complete.