CHAPTER 1

Locally Compact Groups

In this chapter the existence and uniqueness of a left invariant integral is presented. In the first section we develop an averaging procedure that's a functional version of Haar's original set theoretic procedure; we follow this with Andre Weil's proof of existence using Tychonoff's theorem. The second section builds on the averaging procedures broached in the first section. We present an amazing approximation theorem due to Henri Cartan. We follow this with Cartan's Axiom-of-Choice-free proof of the existence of a left invariant integral. Our fourth section presents Cartan's proof of uniqueness, again without use of any transfinite axioms. As usual, we close with a section containing comments about related material and interesting tidbits that complement the earlier presentations.

1. Weil's Proof of Existence

In this short section we present Andre Weil's proof of the existence of a left Haar measure for any topological group. The initial stages of the proof are combinatorial averaging properties comparing continuous functions having compact support. Those these properties are uncomplicated and, once formatted, easily established, they are nonetheless basic to establishing the existence of a Haar measure in a general locally compact group.

Throughout G is a locally compact topological group. Let $\mathcal{K}(G)$ denote the linear space of continuous functions $f: G \to \mathbb{R}$ with compact support, and let $\mathcal{K}^+(G)$ denote those non-zero $f \in \mathcal{K}(G)$ with non-negative values. **Fix, once and for the rest of this discussion, $\omega \in \mathcal{K}^+(G)$.**

Note that if $f, \phi \in \mathcal{K}^+(G)$ then there are numbers $c_1, \ldots, c_n \geq 0$ as well as $g_1, \ldots, g_n \in G$ so that

$$f \leq \sum_{i=1}^{n} c_i {}_{g_i}\phi, \tag{1.1}$$

where ${}_{g_i}\phi(g) = \phi(g_i^{-1}g)$, **denotes a left translation of ϕ.** To see why (1.1) holds, let K be a compact subset of G such that $f(g) = 0$ if $g \notin K$, and let $M > \max\{f(g) : g \in K\}$. Since $\phi \neq 0$ there exists a $g_0 \in G$ for which $\phi(g_0) > 0$; by ϕ's continuity there is an open set U_0 so $g_0 \in U_0$ and $\phi(u) > \frac{\phi(g_0)}{2}$ for all $u \in U_0$. For each $h \in G$, hU_0 is an open set and

$$K \subseteq \bigcup_{h \in G} hU_0;$$

K's compactness leads us to $h_1, \ldots, h_n \in G$ so

$$K = h_1U_0 \cup \cdots \cup h_nU_0.$$

Letting ${}_h\phi(g) = \phi(h^{-1}g)$, we see that if $g \in K$ then $g \in h_m U_0$ for some $1 \le m \le n$ and $h_m^{-1}g \in U_0$ so

$$ {}_{h_m}\phi(g) = \phi(h_m^{-1}g) > \frac{\phi(g_0)}{2}; $$

hence

$$ \frac{2M}{\phi(g_0)}{}_{h_m}\phi(g) > M > f(g), $$

for all $g \in G$ and (1.1) follows.

With this in mind we define

$$ \boldsymbol{(f:\phi) = \inf\left\{\sum_{i=1}^{n} c_i : f \le \sum_{i=1}^{n} c_i\ {}_{g_i}\phi, \ \ \text{for some}\ \ c_1,\dots,c_n \ge 0, g_1,\dots g_n \in G\right\},} $$

and $(f:\phi)$ is the *functional covering number* of f via translates of ϕ.

Lemma 1.1. *For any $f, g, \phi, \psi \in \mathcal{K}^+(G)$ we have*

(i) $0 < (f:\phi) < \infty$;
(ii) *if $h \in G$ then $(f:\phi) = ({}_hf:\phi)$;*
(iii) *$(f:\phi)$ is subadditive and positively homogeneous in f; (i.e., $(f+g:\phi) \le (f:\phi)+(g:\phi)$ and $(\lambda f:\phi) = \lambda(f:\phi)$ for any $\lambda \ge 0$.)*
(iv) *if $f \le g$ then $(f:\phi) \le (g:\phi)$;*
(v) *$(f:\phi) \le (f:\psi)(\psi:\phi)$.*

Proof. All of the statements follow easily using the definition of $(f:\phi)$. We comment on (v): to show $(f:\phi) \le (f:\psi)(\psi:\phi)$, fix $\epsilon > 0$, and let $c_1,\dots,c_n \ge 0$ and $g_1,\dots,g_n \in G$ be chosen so that

$$ f \le \sum_{i=1}^{n} c_{i\,g_i}\psi, $$

with $(f:\psi) \le \sum_{i=1}^n c_i \le (f:\psi)+\epsilon$ and $d_1,\dots,d_m \ge 0$ and $h_1,\dots h_m \in G$ be chosen so that

$$ \psi \le \sum_{j=1}^{m} d_{j\,h_j}\phi, $$

with $(\psi:\phi) \le \sum_{j=1}^m d_j \le (\psi:\phi)+\epsilon$.

It follows that

$$ \begin{aligned} f &\le \sum_{i=1}^{n} c_{i\,g_i}\psi \\ &\le \sum_{i=1}^{n} c_i \ {}_{g_i}\!\left(\sum_{j=1}^{m} d_{j\,h_j}\phi\right) \\ &= \sum_{i=1}^{n} c_i \sum_{j=1}^{m} d_{j\,g_i h_j}\phi, \end{aligned} $$

which, since

$$ {}_{g_i h_j}\phi(g) = {}_{g_i}\phi(h_j^{-1}g) = \phi(g_i^{-1}h_j^{-1}g) = {}_{h_j g_i}\phi(g) $$

says

$$(f : \phi) \leq \sum_{i=1}^{n} c_i \sum_{j=1}^{m} d_j \leq ((f : \psi) + \epsilon)((\psi : \phi) + \epsilon).$$

Let $\epsilon \searrow 0$. Then

$$(f : \phi) \leq (f : \psi)(\psi : \phi). \quad \square$$

For any $f, \phi \in \mathcal{K}^+(G)$ we define

$$\boldsymbol{\mu_\phi(f) = \frac{(f : \phi)}{(\omega : \phi)}}.$$

We'll see that $\mu_\phi(f)$ is an approximation to our ultimate $\int f \, d\mu$; as we squeeze the support of ϕ to ever smaller sets, we'll get better and better approximations. At least, that is the game plan.

LEMMA 1.2. *For any $f, g, \phi \in \mathcal{K}^+(G)$*

(i) $\mu_\phi(f) > 0$;
(ii) $\mu_\phi({}_g f) = \mu_\phi(f)$ *for all* $g \in G$;
(iii) μ_ϕ *is subadditive and positively homogeneous; (i.e.,* $\mu_\phi(f + g) \leq \mu_\phi(f) + \mu_\phi(g)$ *and* $\mu_\phi(\lambda f) = \lambda \mu_\phi(f)$ *for all* $\lambda \geq 0$.*)*
(iv) μ_ϕ *is monotone non-decreasing;*
(v) $\dfrac{1}{(\omega : f)} \leq \mu_\phi(f) \leq (f : \omega)$.

PROOF. We comment on (v): ω, f and ϕ are all non-zero so quantities involved in the inequalities from (v) of Lemma 1.1 are positive. Therefore

$$(f : \phi) \leq (f : \omega)(\omega : \phi),$$

and

$$(\omega : \phi) \leq (\omega : f)(f : \phi);$$

and (v) of Lemma 1.2 is immediate from these. $\square$

Recall that any $f \in \mathcal{K}^+(G)$ is both left- and right- uniformly continuous.

LEMMA 1.3. *Let $f_1, f_2 \in \mathcal{K}^+(G)$ and fix $\epsilon > 0$. Then there exists a neighborhood V of the identity so that if $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V then*

$$\mu_\phi(f_1) + \mu_\phi(f_2) \leq \mu_\phi(f_1 + f_2) + \epsilon.$$

In effect (by Lemma 1.2), as we squeeze the support of ϕ towards $\{e\}$ the function μ_ϕ tends to being additive.

PROOF. Let K be a compact subset of G so that K contains the support of f_1 and the support of f_2. Choose $f \in \mathcal{K}^+(G)$ so that

$$\chi_K \leq f \leq 1.$$

Fix $\delta > 0$ and consider

$$f_\delta = f_1 + f_2 + \delta f.$$

Define h_1, h_2 by

$$h_1(g) = \begin{cases} \frac{f_1(g)}{f_\delta(g)} & \text{if } g \notin K \\ 0 & \text{if } g \in K, \end{cases} \qquad h_2(g) = \begin{cases} \frac{f_2(g)}{f_\delta(g)} & \text{if } g \in K \\ 0 & \text{if } g \notin K. \end{cases}$$

Then $h_1, h_2 \in \mathcal{K}^+(G)$ and each has its support inside that of f_δ. So

$$f_1 = h_1 f_\delta, \quad f_2 = h_2 f_\delta, \quad h_1 + h_2 \le 1.$$

Because any member of $\mathcal{K}^+(G)$ is (left) uniformly continuous on G given $\epsilon' > 0$ there is an open set V containing the identity so that if $x^{-1}y \in V$ then

$$|h_1(x) - h_1(y)| \le \epsilon', \text{ and } |h_2(x) - h_2(y)| \le \epsilon'.$$

Suppose $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V, and suppose

$$f_\delta \le \sum_{i=1}^n c_{i\,g_i}\phi,$$

for some $c_1, \ldots, c_n \ge 0$ and $g_1, \ldots g_n \in V$. Then for any g with $g_i^{-1}g \in V$

$$f_1(g) = f_\delta(g)h_1(g) \le \sum_{i=1}^n c_i\phi(g_i^{-1}g)h_1(g) \le \sum_{i=1}^n c_i\phi(g_i^{-1}g)(h_1(g_i) + \epsilon')$$

and

$$f_2(g) = f_\delta(g)h_2(g) \le \sum_{i=1}^n c_i\phi(g_i^{-1}g)h_2(g) \le \sum_{i=1}^n c_i\phi(g_i^{-1}g)(h_2(g_i) + \epsilon').$$

Since V contains the support of ϕ, it follows that

$$(f_1 : \phi) \le \sum_{i=1}^n c_i(h_1(g_i) + \epsilon'), \text{ and } (f_2 : \phi) \le \sum_{i=1}^n c_i(h_2(g_i) + \epsilon').$$

Adding these together (and remembering that $h_1 + h_2 \le 1$) gives

$$(f_1 : \phi) + (f_2 : \phi) \le \sum_{i=1}^n c_i(1 + 2\epsilon'),$$

and so by the arbitrariness of $c_1, \ldots, c_n, g_1, \ldots, g_n$, with respect to f_δ we see (by the definition of $(f_\delta : \phi)$) that

$$(f_1 : \phi) + (f_2 : \phi) \le (1 + 2\epsilon')(f_\delta : \phi).$$

We divide everything in the inequality by $(\omega : \phi)$, a positive number for sure, and get

$$\begin{aligned} \mu_\phi(f_1) + \mu_\phi(f_2) &\le (1 + 2\epsilon')\mu_\phi(f_\delta) = (1 + 2\epsilon')\mu_\phi(f_1 + f_2 + \delta f) \\ &\le (1 + 2\epsilon')(\mu_\phi(f_1 + f_2) + \delta\mu_\phi(f)) \\ &= \mu_\phi(f_1 + f_2) + 2\epsilon'\mu_\phi(f_1 + f_2) + (1 + 2\epsilon')\delta\mu_\phi(f). \end{aligned}$$

An elementary application of ϵpsilonics will finish the proof of Lemma 1.3. □

All the ingredients are present to proceed to a proof of the existence of Haar Measure on G if we follow A. Weil and call on transfinite methods.

In fact we can use $\mathcal{K}^+(G)$ as an indexing set and keep in mind that for any $\phi, f \in \mathcal{K}^+(G)$,

$$\frac{1}{(\omega : f)} \le \mu_\phi(f) \le (f : \omega). \tag{1.2}$$

With this in mind, we let $I(f)$ be the closed bounded interval

$$I(f) = \left[\frac{1}{(\omega : f)}, (f : \omega)\right]$$

in $\mathbb{R}$ and call on Tychonoff's Theorem (in its original form) to conclude that

$$I = \prod_{\mathcal{K}^+(G)} I(f)$$

is a compact Hausdorff space. For a fixed $\phi \in \mathcal{K}^+(G)$,

$$(\mu_\phi(f))_{f \in \mathcal{K}^+(G)} \in I,$$

thanks to (1.2). For any open set V in G that contains the identity, let $\mathcal{F}_V$ denote the set of all points $(\mu_\phi(f))_{f \in \mathcal{K}^+(G)}$, where ϕ's support is contained in V.

Each $\mathcal{F}_V$ is non-empty.

If V_1, V_2 and V are open sets each of which contains the identity with $V \subseteq V_1 \cap V_2$ then $\mathcal{F}_V \subseteq \mathcal{F}_{V_1} \cap \mathcal{F}_{V_2}$.

By I's compactness there is a point

$$(\mu(f))_{f \in \mathcal{K}^+(G)} \in \bigcap \overline{\mathcal{F}}_V.$$

What does this mean? Well given $f_1, \dots f_n \in \mathcal{K}^+(G)$, $\epsilon > 0$ and an open set V containing G's identity, there is a $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V so

$$|\mu(f_1) - \mu_\phi(f_1)|, \dots, |\mu(f_n) - \mu_\phi(f_n)| \leq \epsilon.$$

An appeal to Lemma 1.2 and Lemma 1.3 show that

$$\begin{cases} \mu({}_g f) = \mu(f),\ \mu(f_1 + f_2) = \mu(f_1) + \mu(f_2) \\ \mu(\lambda f) = \lambda \mu(f),\ \mu(f) \geq \frac{1}{(\omega : f)} > 0 \end{cases}$$

whenever $f_1, f_2 \in \mathcal{K}^+(G)$ and $\lambda > 0$.

If we define $\mu(0) = 0$ then we have what's needed to define an invariant measure on G.

2. A Remarkable Approximation Theorem of Henri Cartan

Throughout G is a locally compact topological group. Let $\mathcal{K}(G)$ denote the linear space of continuous functions $f : G \to \mathbb{R}$ with compact support, and let $\mathcal{K}^+(G)$ denote those non-zero $f \in \mathcal{K}(G)$ with non-negative values. For $g \in G, f \in \mathcal{K}^+(G)$, the function

$${}_g f(x) := f(g^{-1}x) \in \mathcal{K}^+(G).$$

If $f, \phi \in \mathcal{K}^+(G)$ then we define

$$(f : \phi) = \inf \left\{ \sum_{i=1}^n c_i : f \leq \sum_{i=1}^n c_i \, {}_{g_i}\phi, \text{ for some } c_1, \dots, c_n > 0,\ g_1, \dots, g_n \in G \right\},$$

and $(f:\phi)$ is the *functional covering number* of f via translates of ϕ. Fix $\omega \in \mathcal{K}^+(G)$. Define $\mu_\phi(f)$ by

$$\mu_\phi(f) = \frac{(f:\phi)}{(\omega:\phi)}.$$

Here's what we need to know about μ_ϕ.

LEMMA 1.4. *For any $f, g, \phi \in \mathcal{K}^+(G)$*

(i) $\mu_\phi(f) > 0$;
(ii) $\mu_\phi({}_gf) = \mu_\phi(f)$ *for all* $g \in G$;
(iii) μ_ϕ *is subadditive and positively homogeneous; (i.e.,* $\mu_\phi(f+g) \leq \mu_\phi(f) + \mu_\phi(g)$ *and* $\mu_\phi(\lambda f) = \lambda \mu_\phi(f)$ *for all* $\lambda \geq 0$.*)*
(iv) μ_ϕ *is monotone non-decreasing;*
(v) $\dfrac{1}{(\omega:f)} \leq \mu_\phi(f) \leq (f:\omega)$.

THEOREM 1.5 (Bochner-Dieudonné, specially tailored to our purposes). *Let G be a locally compact topological group, and let K be a non-empty compact subset of G and suppose $V_1, \ldots, V_n$ are open subsets of G each with compact closure so that $K \subseteq V_1 \cup \cdots \cup V_n$. Then there are continuous functions $f_1, \ldots, f_n : G \to [0,1]$ such that the support of each f_i is contained in V_i and $\sum_{i=1}^n f_i(k) = 1$ for each $k \in K$.*

The $f_1, \ldots, f_n$ are referred to as *continuous partition of unity subordinate to the cover $\{V_1, \ldots, V_n\}$ of K.*

PROOF. In the "Introduction to Topological Groups" chapter we proved that G is paracompact; hence normal. By G's normality we can find another open cover $\{W_1, \ldots W_n\}$ of K with $\overline{W}_i \subseteq V_i, i = 1, \ldots, n$. (Induction: $n = 1$ follows by the very definition of normality. If we assume that the statement holds for any compact subset of G and any open cover of the set by $m-$open sets for any $m < n$ then let's see what can do if K is a compact set and $K \subseteq V_1 \cup \cdots V_n$. Look at the compact set $C = K \cap (V_2 \cap \cdots V_n)^c \cap \subseteq V_1$. Normality gives us a W_1 so that $C \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1$. Our induction hypothesis applies to

$$K \cap W_1^c \subseteq V_2 \cup \cdots \cup V_n,$$

providing us with the rest of the W_i's, $W_2, \ldots, W_n$.)

Now suppose $K \subseteq V_1 \cup \cdots \cup V_n$; look for $W_1, \ldots, W_n$ so $\overline{W}_i \subseteq V_i$ as in the above. Appeal to Mr. Urysohn and he generously provides us with continuous functions $g_1, \ldots, g_n : G \to [0,1]$ such that $\chi_{\overline{W}_i} \leq g_i \leq \chi_{V_i}$. Set $f_i = \frac{g_i}{\sum_{i=1}^n g_i}$. □

LEMMA 1.6. *Let $f \in \mathcal{K}^+(G)$, K a compact set in G, $\epsilon > 0$ be given. Then there exist $k_1, k_2, \ldots, k_n \in K$, $h_1, \ldots, h_n \in \mathcal{K}^+(G)$ so that if $g \in G, k \in K$ then*

$$\left| \sum_{i=1}^n h_i(k)\, {}_{k_i}f(g) - {}_kf(g) \right| \leq \epsilon.$$

PROOF. Let V be a neighborhood of the identity chosen so that if $g_0 \in Vg$ then

$$|f(g) - f(g_0)| \leq \epsilon,$$

remembering that $f \in \mathcal{K}^+(G)$ must be (right) uniformly continuous on G. Now cover K by a finite collection $k_1V, \ldots, k_nV$ of translates of V where $k_1, \ldots, k_n \in K$. Next, choose a partition of unity $h_1, \ldots, h_n \in \mathcal{K}^+(G)$ subordinate to the covering $\{k_1V, \ldots, k_nV\}$ of K so that

- each $h_i = 0$ outside of k_iV, and
- $\sum_{i=1}^n h_i(k) = 1$ for each and every $k \in K$.

This we can do thanks to the Bochner-Dieudonné partition of unity theorem (Theorem 1.5).

Notice that if $k \in V$ and $k \in k_iV$ then

$$(k_i^{-1}g)(k^{-1}g)^{-1} = k_i^{-1}gg^{-1}k = k_i^{-1}k \in V,$$

and so by choice of V,

$$|f(k_i^{-1}g) - f(k^{-1}g)| \leq \epsilon.$$

If $k \in V$ but $k \notin k_iV$ then $h_i(k) = 0$. In either case

$$|h_i(k)_{k_i}f(g) - h_i(k)_kf(g)| \leq \epsilon h_i(k)$$

and this is so for $i = 1, \ldots, n$. Adding these inequalities up and recalling that $\sum_{i=1}^n h_i(k) = 1$ for each $k \in K$, we get

$$\left| \sum_{i=1}^n h_i(k)_{k_i}f(g) - {}_kf(g) \right| \leq \epsilon$$

for all $k \in K$ and all $g \in G$, as advertised. □

Lemma 1.7. *Let $f_1, f_2, \ldots, f_n \in \mathcal{K}^+(G), 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n$, and $\epsilon > 0$ be given. Then there is an open set V containing the identity of G so that if $\psi \in \mathcal{K}^+(G)$ with the support of ψ contained in V then*

$$\sum_{i=1}^n \lambda_i \mu_\psi(f_i) \leq \mu_\psi\left(\sum_{i=1}^n \lambda_i f_i\right) + \epsilon.$$

Proof. Let K be a compact set that contains the support of $f_1, \ldots, f_n$. Choose $f \in \mathcal{K}^+(G)$ so that

$$\chi_K \leq f \leq 1.$$

Let $\delta > 0$. Look at

$$f_\delta = \lambda_1 f_1 + \cdots + \lambda_n f_n + \delta f \in \mathcal{K}^+(G).$$

Define $h_i, 1 \leq i \leq n$ by

$$h_i(g) = \begin{cases} \frac{\lambda_i f_i(g)}{f_\delta(g)} & \text{if } g \in K \\ 0 & \text{if } g \notin K. \end{cases}$$

Notice $\lambda_i f_i = f_\delta h_i$ and $\sum_{i=1}^n h_i \leq 1$. Moreover $h_i \in \mathcal{K}^+(G)$. The uniform continuity of members of $\mathcal{K}^+(G)$ ensures that if $\epsilon' > 0$ is given then there is an open set V that contains the identity so that if $y \in xV$ then

$$|h_1(x) - h_1(y)|, \ldots, |h_n(x) - h_n(y)| \leq \epsilon'.$$

Suppose $\psi \in \mathcal{K}^+(G)$ has its support contained in V and suppose $c_1, \ldots, c_m > 0$ and $g_1, \ldots, g_m \in V$ are chosen so that

$$\lambda_1 f_1 + \cdots + \lambda_n f_n + \delta f = f_\delta \leq \sum_{j=1}^n c_{j\,g_j}\psi.$$

If $g \in g_j V$ then for each $i = 1, \ldots, n$

$$\begin{aligned} \lambda_i f_i(g) &= f_\delta h_i(g) \le \sum_{j=1}^{m} c_j \psi(g_j^{-1} g) h_i(g) \\ &\le \sum_{j=1}^{m} c_j \psi(g_j^{-1} g)(h_i(g_j) + \epsilon^{'}). \end{aligned}$$

It follows that for $i = 1, \ldots, n$

$$\lambda_i(f_i : \psi) = (\lambda_i f_i : \psi) \le \sum_{j=1}^{m} c_j (h_i(g_j) + \epsilon^{'}),$$

and so keeping in mind the fact that $\sum_{i=1}^{n} h_i \le 1$,

$$\sum_{i=1}^{n} \lambda_i (f_i : \psi) = \sum_{i=1}^{n} (\lambda_i f_i : \psi) \le \sum_{j=1}^{m} c_j (1 + n\epsilon^{'}).$$

By the arbitrariness of $c_1, \ldots, c_m > 0$ and $g_1, \ldots, g_m \in V$ vis-a-vis f_δ we see that

$$\sum_{i=1}^{n} \lambda_i (f_i : \psi) \le (1 + n\epsilon^{'})(f_\delta : \psi).$$

On dividing everything in plain sight by $(\omega : \psi)$ we get

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i \mu_\psi(f_i) &\le (1 + n\epsilon^{'})(\mu_\psi(f_\delta)) \\ &= (1 + n\epsilon^{'})(\mu_\psi(\lambda_1 f_1 + \cdots + \lambda_n f_n + \delta f)) \\ &\le (1 + n\epsilon^{'})(\mu_\psi(\lambda_1 f_1 + \cdots + \lambda_n f_n) + \mu_\psi(\delta f)) \\ &= \mu_\psi(\lambda_1 f_1 + \cdots + \lambda_n f_n) + n\epsilon^{'} \mu_\psi(\lambda_1 f_1 + \cdots + \lambda_n f_n) + \delta(1 + n\epsilon^{'})\mu_\psi(f). \end{aligned}$$

The standard dose of ϵpsilonics finishes the proof. □

A key element in the proof of the existence (and uniqueness) of Haar measure is the following stunning approximation theorem of H. Cartan. Remarkably, there is no measure theory in the statement of the theorem.

THEOREM 1.8 (Cartan). *Let $f \in \mathcal{K}^+(G)$, and let $0 < \epsilon$ be given. Then there is an open set V in G that contains the identity, has compact closure such that if $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V, then there exist $g_1, \ldots, g_n \in \overline{supp}(f), c_1, \ldots, c_n \ge 0$ so that*

$$\left\| f - \sum_{i=1}^{n} c_{i_{g_i}} \phi \right\| \le \epsilon.$$

PROOF. To start, let $0 < \epsilon^{'} < \epsilon$ and choose an open set V with $\overline{V}$ compact, $e \in V$ so that $|f(g) - f(g_0)| \le \epsilon^{'}$ whenever $g^{-1} g_0 \in V$.

Let K be a compact set containing the support of f. Let $\delta > 0$. Apply Lemma 1.6 to find $g_1, \ldots, g_n \in K$ and $h_1, \ldots, h_n \in \mathcal{K}^+(G)$ so that if $g \in G, k \in K$ then

$$\left| \sum_{i=1}^{n} h_i(k)_{g_i}\phi(g) - {}_k\phi(g) \right| \leq \delta.$$

Multiplying by $f(k)$, we see that for all $g \in G, k \in K$, we have

$$\left| \sum_{i=1}^{n} {}_{g_i}\phi(g) h_i(k) f(k) - {}_k\phi(g) f(k) \right| \leq \delta f(k); \tag{1.3}$$

but the support of f is contained in K so if $k \notin K$ the LHS of (1.3) is zero and so we have (1.3) holding for all $k, g \in G$.

Now ϕ's support lies in V, so if $k^{-1}g \in V$ we see

$$|f(g) - f(k)| \leq \epsilon^{'},$$

and so

$$|{}_k\phi(g) f(g) - {}_k\phi(g) f(k)| \leq \epsilon^{'}{}_k\phi(g);$$

On the other hand if $k^{-1}g \notin V$ then ${}_k\phi(g) = \phi(k^{-1}g) = 0$. Regardless of where $k^{-1}g$ is located, we have that for every $k \in G$ and $g \in G$

$$|{}_k\phi(g) f(k) - {}_k\phi(g) f(g)| \leq \epsilon^{'}{}_k\phi(g). \tag{1.4}$$

In tandem, (1.3) and (1.4) tell us that for any $k, g \in G$

$$\begin{aligned} \left| \sum_{i=1}^{n} {}_{g_i}\phi(g) h_i(k) f(k) - {}_k\phi(g) f(g) \right| &\leq \left| \sum_{i=1}^{n} {}_{g_i}\phi(g) h_i(k) f(k) - {}_k\phi(g) f(k) \right| + |{}_k\phi(g) f(k) - {}_k\phi(g) f(g)| \\ &\leq \delta f(k) + \epsilon^{'}{}_k\phi(g). \end{aligned} \tag{1.5}$$

Notice that if $\check{\phi}(g) = \phi(g^{-1})$ then

$${}_k\phi(g) = \phi(k^{-1}g) = \check{\phi}(g^{-1}k) = {}_g\check{\phi}(k).$$

If we now replace ${}_k\phi(g)$ by ${}_g\check{\phi}(k)$ in (1.5) we get that for all $k, g \in G$,

$$\left| \sum_{i=1}^{n} {}_{g_i}\phi(g) h_i(k) f(k) - {}_g\check{\phi}(k) f(g) \right| \leq \delta f(k) + \epsilon^{'}{}_g\check{\phi}(k). \tag{1.6}$$

Now viewing g as fixed and letting k be free, we get the functional inequality

$$\boldsymbol{\left| \sum_{i=1}^{n} {}_{g_i}\phi(g) h_i f - {}_g\check{\phi} f(g) \right| \leq \delta f + \epsilon^{'}{}_g\check{\phi}.}$$

We now call on the averaging machinery developed earlier.

With the inequality

$$\left| \sum_{i=1}^{n} {}_{g_i}\phi(g) h_i f - {}_g\check{\phi} f(g) \right| \leq \delta f + \epsilon^{'}{}_g\check{\phi}$$

in hand, note that for $\psi \in \mathcal{K}^+(G)$, the subadditivity of the functional μ_ψ tells us that

$$\left|\mu_\psi\left(\sum_{i=1}^n {}_{g_i}\phi(g)h_i f\right) - f(g)\mu_\psi({}_g\check{\phi})\right| \le \delta\mu_\psi(f) + \epsilon^{'}\mu_\psi({}_g\check{\phi}).$$

Now $\phi \neq 0$ and by Lemma 1.2, $\mu_\psi({}_g\check{\phi}) = \mu_\psi(\check{\phi}) > 0$ so division by $\mu_\psi({}_g\check{\phi})$ is legal (if not ethical). Do it! The result:

$$\left|\mu_\psi\left(\sum_{i=1}^n \frac{{}_{g_i}\phi(g)}{\mu_\psi({}_g\check{\phi})}h_i f\right) - f(g)\right| \le \delta\frac{\mu_\psi(f)}{\mu_\psi(\check{\phi})} + \epsilon^{'}. \tag{1.7}$$

This is so for any $\psi \in \mathcal{K}^+(G)$.

But Lemma 1.6 says that given $\eta > 0$ there is an open set W containing the identity of G so that if $\psi \in \mathcal{K}^+(G)$ and the support of ψ is inside W then (setting $\lambda_i = \frac{{}_{g_i}\phi(g)}{\mu_\psi(\check{\phi})}$)

$$\left|\mu_\psi\left(\sum_{i=1}^n \frac{{}_{g_i}\phi(g)}{\mu_\psi(\check{\phi})}h_i f\right) - \sum_{i=1}^n \frac{{}_{g_i}\phi(g)}{\mu_\psi(\check{\phi})}\mu_\psi(h_i f)\right| \le \eta. \tag{1.8}$$

Note: W acts as a catalyst ensuring near additivity of μ_ψ but also providing a term

$$\mu_\psi\left(\sum_{i=1}^n \frac{{}_{g_i}\phi(g)}{\mu_\psi(\check{\phi})}h_i f\right)$$

to compare with both

$$f(g) \text{ and a sum of the sort } \sum_{i=1}^n c_{i\,g_i}\phi(g).$$

We're in business! Let

$$c_i = \frac{\mu_\psi(h_i f)}{\mu_\psi(\check{\phi})}$$

then each $c_i > 0$ and for any $g \in G$

$$\begin{aligned}
\left|f(g) - \sum_{i=1}^n c_{i\,g_i}\phi(g)\right| &= \left|f(g) - \sum_{i=1}^n \frac{\mu_\psi(h_i f)}{\mu_\psi(\check{\phi})}\,{}_{g_i}\phi(g)\right| \\
&\le \left|f(g) - \mu_\psi\left(\sum_{i=1}^n \frac{{}_{g_i}\phi(g)}{\mu_\psi(\check{\phi})}h_i f\right)\right| + \left|\mu_\psi\left(\sum_{i=1}^n \frac{{}_{g_i}\phi(g)}{\mu_\psi(\check{\phi})}h_i f\right) - \sum_{i=1}^n \frac{{}_{g_i}\phi(g)}{\mu_\psi(\check{\phi})}\mu_\psi(h_i f)\right| \\
&\le \delta\frac{\mu_\psi(f)}{\mu_\psi(\check{\phi})} + \epsilon^{'} + \eta \quad \text{(by (1.7) and (1.8))}.
\end{aligned}$$

Standard ϵpsilonics take over to finish the proof. □

3. Cartan's Proof of Existence of a Left Haar Integral

Before presenting Cartan's proof of the existence of a left invariant Haar integral we'll state a crucial lemma. We delay the proof of the lemma because it's also critical to Cartan's proof of uniqueness and the pattern of its proof is worthy of close inspection.

Lemma 1.9. *Let $f \in \mathcal{K}^+(G)$ and $\epsilon > 0$ be given. Then there is an open set U containing G's identity for which: given any $h \in \mathcal{K}^+(G)$ with the support of h contained in U there is $c \geq 0$ and an open set $V = V_U$ containing the identity so that*

$$|\mu_\phi(f) - c\mu_\phi(h)| \leq \epsilon$$

whenever $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V.

The existence of a left Haar integral follows from our next result.

Theorem 1.10. *Let $f \in \mathcal{K}^+(G), \epsilon > 0$. Then there is an open set V containing G's identity for which*

$$|\mu_\phi(f) - \mu_\psi(f)| < \epsilon$$

whenever $\phi, \psi \in \mathcal{K}^+(G)$ with the support of ϕ and the support of ψ contained in V.

Proof. Suppose $0 < \delta \leq 1/7$. Then according to the dictates of our previous lemma (Lemma 1.9) that there is an open set U containing G's identity so that for any $h \in \mathcal{K}^+(G)$ with the support of h contained in U, there is a $c_f \geq 0$ and another open set V_U containing the identity so that

$$|\mu_\phi(f) - c_f\mu_\phi(h)| \leq \delta, \tag{1.9}$$

whenever $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V_U.

If we apply our lemma again (Lemma 1.9) with our beloved ω replacing f and the same δ in place, we find an open set $\widetilde{U}$ containing the identity so that if $h \in \mathcal{K}^+(G)$ with the support of h contained in $\widetilde{U}$ then for some $c_\omega \geq 0$ and some open set $\widetilde{V}_{\widetilde{U}}$ containing the identity, we have

$$|1 - c_\omega\mu_\phi(h)| = |\mu_\phi(\omega) - c_\omega\mu_\phi(h)| \leq \delta \tag{1.10}$$

whenever $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in $\widetilde{V}_{\widetilde{U}}$.

On replacing $U, \widetilde{U}$ by $U \cap \widetilde{U}$ and $V_U, \widetilde{V}_{\widetilde{U}}$ by $V_U \cap \widetilde{V}_{\widetilde{U}}$, we can and do suppose that the same U, V work in both (1.9) and (1.10).

Notice that $\delta < 1$, so (1.10) ensures us that $c_\omega > 0$ making division by c_ω a legal operation. Let

$$R = \frac{c_f}{c_\omega}. \tag{1.11}$$

If $\phi \in \mathcal{K}^+(G)$ with the support of ϕ is contained in V then both (1.9) and (1.10) are in effect. So

$$\begin{aligned}
|\mu_\phi(f) - R| &= |\mu_\phi(f) - c_f\mu_\phi(h) + c_f\mu_\phi(h) - R| \\
&\leq |\mu_\phi(f) - c_f\mu_\phi(h)| + c_f\left|\mu_\phi(h) - \frac{1}{c_\omega}\right| \\
&\leq \delta + \frac{c_f}{c_\omega}|c_\omega\mu_\phi(h) - 1| \\
&\leq \delta + R\delta = (1+R)\delta.
\end{aligned}$$

Appealing again to (1.9) we see that

$$c_f\mu_\phi(h) \leq \mu_\phi(f) + \delta = \frac{(f:\phi)}{(\omega:\phi)} + \delta \leq (f:\omega) + \delta \tag{1.12}$$

because $(f:\phi) \le (f:\omega)(\omega:\phi)$. By (1.10) we have

$$c_\omega \mu_\phi(h) \ge 1-\delta. \tag{1.13}$$

Since $\mu_\phi(h) > 0$ we see (using $\delta \le 1/2$) that

$$R = \frac{c_f}{c_\omega} = \frac{c_h\mu_\phi(h)}{c_\omega\mu_\phi(h)} \le \frac{(f:\omega)+\delta}{1-\delta} \le 2(f:\omega)+1.$$

Investing this knowledge wisely we find

$$|\mu_\phi(f) - R| \le (1+R)\delta \le (1+2(f:\omega)+1)\delta = 2(1+(f:\omega))\delta$$

so long as $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V.

Cutting to the chase: given $f \in \mathcal{K}^+(G)$ and $\epsilon > 0$ there is an $R \ge 0$ and an open set V containing G's identity so that

$$|\mu_\phi(f) - R| \le \epsilon$$

whenever $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V. Of course if $\psi \in \mathcal{K}^+(G)$ has its support inside V then

$$|\mu_\psi(f) - R| \le \epsilon$$

too.

Conclusion: given $f \in \mathcal{K}^+(G)$ and $\epsilon > 0$ there is an open set V containing the identity so that if $\phi, \psi \in \mathcal{K}^+(G)$ have their support inside V then

$$|\mu_\phi(f) - \mu_\psi(f)| \le 2\epsilon.$$

Among friends, this is good enough to finish the proof. □

Our hard work is done. It's time to reap the rewards.

Existence of a Left Invariant Integral: We now look at all pairs (V,ϕ) where V is an open set in G containing G's identity and $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V. We order the pairs (V,ϕ) by

$$(V_1,\phi_1) \le (V_2,\phi_2)$$

if $V_2 \subseteq V_1$; so convergence with respect to this direction measures what happens as that V coordinates shrinks to $\{e\}$.

Fixing $f \in \mathcal{K}^+(G)$ we see that our last theorem tells us that

$$\mu(f) \equiv \lim_{(V,\phi)} \mu_\phi(f)$$

exists. By Lemma 1.3, the functional μ is additive on $\mathcal{K}^+(G)$; Lemma 1.2 tells us that μ is positively homogeneous and invariant under left translates. What's more, for any $\phi \in \mathcal{K}^+(G)$

$$\mu_\phi(\omega) = \frac{(\omega:\phi)}{(\omega:\phi)} = 1$$

so $\mu(\omega) = 1$. Therefore μ is a left invariant Haar measure on G.

4. Cartan's Proof of Uniqueness

We restate Lemma 1.9 and include its proof.

LEMMA 1.11. *Let $f \in \mathcal{K}^+(G)$ and $\epsilon > 0$ be given. Then there is an open set U containing G's identity for which: given any $h \in \mathcal{K}^+(G)$ with the support of h contained in U there is $c \geq 0$ and an open set $V = V_U$ containing the identity so that*

$$|\mu_\phi(f) - c\mu_\phi(h)| \leq \epsilon$$

whenever $\phi \in \mathcal{K}^+(G)$ with the support of ϕ contained in V.

PROOF. Let W be an open set containing the identity with $\overline{W}$ compact, and let $K = \overline{\{x : f(x) \neq 0\}}$ be the support of f. Fix $f' \in \mathcal{K}^+(G)$ so that $f' \geq \chi_{K \cdot W}$.

For any $\eta > 0$, we know (by Cartan's approximation theorem) that there is an open set U containing the identity (we can suppose that $U \subseteq W$) so that if $h \in \mathcal{K}^+(G)$ with the support of h contained in U then there are $k_1, \ldots, k_n \in K$ and $c_1, \ldots, c_n \geq 0$ so that

$$\left\| f - \sum_{i=1}^{n} c_i \, {}_{k_i}h \right\| \leq \eta.$$

Notice that h vanishes outside of U and so outside of W. It soon follows that each of the functions ${}_{k_i}h$ vanish outside k_iW and so outside $K \cdot W$. But f itself vanishes outside K which lies inside $K \cdot W$. So f and ${}_{k_i}h$ both vanish outside $K \cdot W$. Using this and the behaviour of f' we see that we can upgrade the above inequality to

$$\left| f - \sum_{i=1}^{n} c_i \, {}_{k_i}h \right| \leq \eta f',$$

a functional inequality involving members of $\mathcal{K}^+(G)$. Now if $\phi \in \mathcal{K}^+(G)$ we see that

$$\left| \mu_\phi(f) - \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) \right| \leq \eta \mu_\phi(f') \leq \eta (f' : \omega) \tag{1.14}$$

(the last inequality follows by Lemma 1.2(v)).

An appeal to our new additive lemma (Lemma 1.7) tells us that there is an open set $V = V_U$ containing the identity so that if we further stipulate that ϕ have its support contained in V then we can assure that

$$\left| \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) - \sum_{i=1}^{n} c_i \mu_\phi({}_{k_i}h) \right| \leq \eta. \tag{1.15}$$

But $\mu_\phi({}_{k_i}h) = \mu_\phi(h)$ so combining (1.14) and (1.15) and letting $c = \sum_{i=1}^{n} c_i$

$$\begin{aligned} |\mu_\phi(f) - c\mu_\phi(h)| &= \left| \mu_\phi(f) - \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) + \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) - \sum_{i=1}^{n} c_i \mu_\phi(h) \right| \\ &\leq \left| \mu_\phi(f) - \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) \right| + \left| \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) - \sum_{i=1}^{n} c_i \mu_\phi(h) \right| \\ &\leq \eta(f' : \omega) + \eta. \end{aligned}$$

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ϵpsilonics now rule! □

Uniqueness of μ: To prove the uniqueness of left invariant integrals on $\mathcal{K}^+(G)$, we consider a functional ν on $\mathcal{K}^+(G)$ which satisfies $\nu(f) > 0$ for each $f \in \mathcal{K}^+(G)$, ν is additive and positively homogeneous and ν is left invariant so $\nu(f) = \nu({}_g f)$ for any $g \in G$. We further assume that $\nu(\omega) = 1$, where ω is our chosen one in $\mathcal{K}^+(G)$. What do we want? We want to show that ν is in fact, $\lim_\phi \mu_\phi(f)$.

The path blazed by Cartan leads through the proof of Lemma 1.11. Let us repeat the lemma's proof but with ν in mind instead of μ_ϕ. Let $f \in \mathcal{K}^+(G)$.

We start with an open set W containing the identity and having compact closure, $\overline{W}$. Let $K = \overline{\{x : f(x) \neq 0\}}$ be the support of f. Pick $f' \in \mathcal{K}^+(G)$ so that $f' \geq \chi_{K \cdot W}$.

Exactly as before we know that given $\eta > 0$ there is an open set U containing the identity (contained, if we wish - and we do - inside W) so that if $h \in \mathcal{K}^+(G)$ with the support of h contained in U then there are $k_1, \ldots, k_n \in K$ and $c_1, \ldots, c_n \geq 0$ so that

$$\left\| f - \sum_{i=1}^{n} c_i \, {}_{k_i}h \right\| \leq \eta.$$

As before, analysis of the supports of f and h soon tell us that

$$\left| f - \sum_{i=1}^{n} c_i \, {}_{k_i}h \right| \leq \eta f'.$$

Now the inequality (1.14)

$$\left| \mu_\phi(f) - \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) \right| \leq \eta \mu_\phi(f') \leq \eta (f' : \omega)$$

is replaced by

$$\boldsymbol{\left| \nu(f) - \nu\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) \right| \leq \eta \nu(f').}$$

Since ν is additive and positively homogeneous, the inequality (1.15)

$$\left| \mu_\phi\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) - \sum_{i=1}^{n} c_i \mu_\phi({}_{k_i}h) \right| \leq \eta$$

is replaced by the identity

$$\boldsymbol{\nu\left(\sum_{i=1}^{n} c_i \, {}_{k_i}h \right) = \sum_{i=1}^{n} c_i \nu({}_{k_i}h).}$$

The left invariance of ν tells us that $\nu({}_{k_i}h) = \nu(h)$ so letting $c = \sum_{i=1}^{n} c_i$,

$$\begin{aligned} \boldsymbol{|\nu(f) - c\nu(h)|} &= \boldsymbol{\left|\nu(f) - \sum_{i=1}^{n} c_i \nu(h)\right|} \\ &\leq \boldsymbol{\left|\nu(f) - \nu\left(\sum_{i=1}^{n} c_{i k_i} h\right)\right|} \\ &\leq \boldsymbol{\eta\nu(f^{'})}. \end{aligned}$$

Note: The "c'' is the same here as it was in the case of μ_ϕ or at least, it can be chosen to be so.

If we now turn to the proof of the existence of a left invariant integral μ (Theorem 1.10), we can easily see that the constant c_ω found therein also works with ν replacing μ_ϕ, this time without any codicils about ϕ's support. As a result if $R = \frac{c}{c_\omega}$ as in that proof (see (1.11)) we can conclude the same truism: there is an open set U containing the identity so that for any $h \in \mathcal{K}^+(G)$ there is a $c_\omega > 0$ so that

$$|1 - c_\omega \nu(h)| = |\nu(\omega) - c_\omega \nu(h)| \leq \delta$$

so long as h's support is contained in U. The point, once again, is that c_ω can be chosen within the realm of ϵsilonics to be the same for ν as for μ_ϕ; again the roots are in Cartan's approximation theorem. The resulting R for ν is the same as for μ_ϕ and so ultimately, $|\nu(f) - \mu_\phi(f)| \leq 2\epsilon$ just as in the proof of existence.

The upshot: for any $\epsilon > 0$

$$|\nu(f) - \mu_\phi(f)| \leq 2\epsilon.$$

CHAPTER 2

Metric Invariance and Haar Measure

Suppose G is a locally compact metrizable group. Then G admits a metric ρ which is left invariant and generates G's topology; G also admits a left invariant regular Borel measure (a left Haar measure). Is there any connection between these left invariant objects? In this chapter we show the answer is an emphatic "yes."

The highlight of the chapter is the wedding of Haar measure and Hausdorff measure; presiding at the ceremony is C. Bandt. The result is a beautiful offspring: *fractional Hausdorff measure.* A consequence is that if λ is a left Haar measure on G and ρ is a left invariant metric generating G's topology then $\rho-$isometric subsets of G have the same $\lambda-$outer measure.

Now the assumption that G have a left invariant metric that generates its group topology is, of course, not an assumption at all. The Birkhoff-Kakutani Theorem assures us that G's metrizability is enough to ensure such a metric exists. However since we are looking for relationships between left invariant measures on G and left invariant metrics on G, it is instructive (and fun!) to show how to generate a left invariant metric that generates G's topology directly from its left Haar measure. This proof is due to R.A. Struble and leads off the chapter.

THEOREM 2.1 (R.A. Struble). *Let G be a locally compact group with left Haar measure λ. Let (V_n) be a decreasing sequence of open sets that form a neighborhood basis of the identity e in G where $\overline{V}_n$ compact for each n. Then*

$$\rho(x, y) = \sup_n \lambda(xV_n \triangle yV_n)$$

defines a left invariant metric on G which is compatible with the topology of G.

PROOF. It's clear that $\rho(x, y)$ is well-defined and that $\rho(x, y) = \rho(y, x)$. Moreover $\rho(x, y) \geq 0$ and $\rho(x, y) < \infty$ regardless of $x, y \in G$ since each V_n is a Borel set with compact closure. Further, $\rho(zx, zy)$ and $\rho(x, y)$ coincide because λ is left invariant.

If $x \neq y$ then there must be an m so that $xV_m \cap yV_m = \emptyset$ since G is Hausdorff; but now

$$\rho(x, y) \geq \lambda(xV_m \triangle yV_m) = 2\lambda(V_m) > 0.$$

On noticing that for any n and any $x, y, z \in G$,

$$xV_n \triangle yV_n \subseteq (xV_n \triangle zV_n) \cup (zV_n \triangle yV_n),$$

we see that for any $x, y, z \in G$,

$$\begin{aligned} \lambda(xV_n \triangle yV_n) &\leq \lambda((xV_n \triangle zV_n) \cup (zV_n \triangle yV_n)) \\ &\leq \lambda(xV_n \triangle zV_n) + \lambda(zV_n \triangle yV_n) \\ &\leq \rho(x, z) + \rho(z, y), \end{aligned}$$

and with this

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y).$$

In sum, ρ is a left invariant metric on G.

If G's topology is discrete then $\lambda(\{e\}) > 0$ so $V_m = \{e\}$ for some m; hence if $x \neq y$,

$$\rho(x, y) \geq \lambda(xV_m \triangle yV_m) = \lambda(\{x, y\}) = 2\lambda(\{e\}) > 0,$$

and the topology induced by ρ is discrete.

If G's topology is *not* discrete then $\lambda(V_n) \searrow \lambda(\cap_n V_n) = \lambda(\{e\}) = 0$. If V is any open set containing e then there is an $m \in \mathbb{N}$ so that $V_m V_m^{-1} \subseteq V$.

Claim 1: $x \in V$ *whenever* $\rho(x, e) < \lambda(V_m)$. To see this, let $\rho(x, e) < \lambda(V_m)$. Then

$$\lambda(xV_m \triangle V_m) \leq \rho(x, e) < \lambda(V_m),$$

a positive number. Were $xV_m \cap V_m = \emptyset$ then

$$\lambda(xV_m \triangle V_m) = 2\lambda(V_m) < \lambda(V_m),$$

oops! So $xV_m \cap V_m \neq \emptyset$ and thus there are $v_1, v_2 \in V_m$ so $xv_1 = v_2 \in xV_m \cap V_m$ and

$$x = v_2 v_2^{-1} \in V_m V_m^{-1} \subseteq V.$$

This is so whenever $\rho(x, e) < \lambda(V_m)$, and our claim is justified.

Let's look at all of the points x such that $\rho(x, e) < r$, where $r \in \mathbb{Q}, r > 0$. There must be an $m \in \mathbb{N}$ so that $\lambda(V_n) < \frac{r}{4}$, whenever $n \geq m$. Each of the functions

$$f_k(x) = \lambda(xV_k \triangle V_k)$$

is continuous and satisfies $f_k(e) = \lambda(V_k \triangle V_k) = \lambda(\emptyset) = 0$. But now we know there is an $l \in \mathbb{N}$ so that if $x \in V_l$ then $f_1(x), \ldots, f_{m-1}(x) < r$.

Claim 2: *if* $x \in V_l$ *then* $\rho(x, e) < r$. Why is this so? Well if $x \in V_l$ then by choice of $l \in \mathbb{N}$, we have

$$\lambda(xV_1 \triangle V_1), \ldots, \lambda(xV_{m-1} \triangle V_{m-1}) < r.$$

What about $\lambda(xV_k \triangle V_k)$ for $k \geq m$? In this case,

$$\lambda(xV_k \triangle V_k) \leq 2\lambda(V_k) < 2 \cdot \frac{r}{4} < r.$$

It follows that $\rho(x, e) = \sup_n \lambda(xV_m \triangle V_n) < r$.

Our two claims taken in tandem show that ρ generates G's topology about e. Since ρ is left invariant and since G's topology is too, this is enough to say that ρ generates G's topology everywhere. $\square$

THEOREM 2.2 (C. Bandt). *If ρ is a left invariant metric on the locally compact metrizable group G defining the topology of G, then any two subsets of G that are ρ-isometric have the same left Haar measure.*

To prove Theorem 2.2 we need to develop fractional Hausdorff measure. let A be a fixed compact set with non-empty interior;

We'd like to construct a Hausdorff gauge function h so that the associated Hausdorff measure μ^h on G satisfies

$$0 < \mu^h(A) < \infty.$$

Sadly finding such a gauge function is elusive. Happily Bandt found a way around this: fractional Hausdorff measure. Given a Hausdorff gauge function h we define the **fractional Hausdorff measure** ν^h by

$$\nu^h(E) = \lim_{t \searrow 0} \inf \left\{ \sum_j c_j h(\mathrm{diam}(B_j)) : c_j > 0, \chi_E \leq \sum_j c_j \chi_{B_j}, \mathrm{diam}(B_j) \leq t \right\}.$$

Mimicking the proofs encountered in Hausdorff measures, we see that ν^h is a metric outer measure (ensuring us that Borel sets are ν^h-measurable) and ν^h is left invariant reflecting ρ's left invariance. The issue is to judiciously choose h so that

$$0 < \nu^h(A) < \infty.$$

Once this has been achieved, we see that for any compact set $K \subseteq G, \nu^h(K) < \infty$ (K can be covered by finitely many of A's left translates, each of which has the same ν^h-measure) and for any non-empty open subset U of G, $\nu^h(U) > 0$ (we can cover A by finitely many left translates of U, each having the same ν^h-measure).

Before proceeding we take note that if $0 < s < t$ then for any $E \subseteq G$

$$\inf \left\{ \sum_j c_j h(\mathrm{diam}(B_j)) : c_j > 0, \chi_E \leq \sum_j c_j \chi_{B_j}, \mathrm{diam}(B_j) \leq t \right\} \tag{2.1}$$

is at least as large

$$\inf \left\{ \sum_j c_j h(\mathrm{diam}(B_j)) : c_j > 0, \chi_E \leq \sum_j c_j \chi_{B_j}, \mathrm{diam}(B_j) \leq s \right\};$$

after all, there are at least as many fractional coverings of E when the sets B_j have diameter less than or equal to s as there are with sets B_j having diameter less than or equal to t. So, as function of t, (2.1) ascends as t descends to zero. It follows that $\nu^h(E)$ *is determined by what happens when the fractional coverings involve sets of small diameter.*

The gauge function that works is

$$h(t) = \sup\{\lambda(B) : \mathrm{diam}(B) := \mathrm{diam}(B) \leq t\},$$

a function that takes finite values on some open interval $(0, T_0)$. Indeed, G is locally compact and so G has a basis for its open sets consisting of sets with compact closure; it follows that at any point of G, we have a basis of open balls with compact closure all with diameter less than T_0, for some $T_0 > 0$. The left invariance of ρ ensures that the same T_0 works throughout G.

NOTE 2.3. *ν^h is left invariant.*

As a matter of fact, the metric ρ of Theorem 2.1 that generates G's topology is left invariant and so for any subset B of G, the diameter of B and gB are the same, since they're $\rho-$isometric.

By the same token, sets in G that are isometric with respect to ρ are assigned the same ν^h-values.

The real issue with ν^h is to show that it's non-trivial, that is, $0 < \nu^h(A) < \infty$. Once we know this to be so then ν^h is a left Haar measure on G and so is but a multiple of λ. Hence sets in G that are isometric (with respect to ρ) have the same $\lambda-$measure as well and we will have proved Theorem 2.2. We proceed with several lemmas to get that ν^h is non-trivial.

LEMMA 2.4. *For any Borel set $B \subseteq G$,*

$$\lambda(B) \leq \nu^h(B).$$

PROOF. Suppose $\chi_B \leq \sum_j c_j \chi_{B_j}$. Then $\chi_B \leq \sum_j c_j \chi_{\overline{B}_j}$ and so

$$\begin{aligned} \lambda(B) &= \int_B d\lambda \leq \int \sum c_j \chi_{\overline{B}_j}\, d\lambda \\ &= \sum c_j \lambda(\overline{B}_j) \leq \sum c_j h(\text{diam}(\overline{B}_j)) \\ &= \sum c_j h(\text{diam}(B_j)). \end{aligned}$$

It follows that

$$\lambda(B) \leq \nu^h(B). \quad \square$$

Of course a particular consequence of this lemma is

COROLLARY 2.5. $0 < \lambda(A) < \nu^h(A)$.

LEMMA 2.6. *Let $E(t)$ be defined by*

$$E(t) = \inf \left\{ \sum_{j=1}^{n} c_j : n \in \mathbb{N}, \chi_A \leq \sum_{j=1}^{n} c_j \chi_{B_j}, c_j \geq 0, diam(B_j) \leq t \right\}.$$

If

$$\liminf_t h(t)E(t) < \infty$$

then $\nu^h(A) < \infty$.

PROOF. There is a $c > 0$ and a sequence (t_k), $t_k > 0$ with $t_k \searrow 0$ so that

$$h(t_k)E(t_k) < c$$

for all k. In other words,

$$E(t_k) < \frac{c}{h(t_k)}$$

for all k. For each k, choose a fractional covering of A,

$$\chi_A \leq \sum_{j=1}^{n(k)} c_j^{(k)} \chi_{B_j^{(k)}}, \quad c_j^{(k)} \geq 0$$

with

$$\sum_{j=1}^{n(k)} c_j^{(k)} \leq \frac{c}{h(t_k)}.$$

Of course, the definition of the fractional Hausdorff measure $\nu^h(A)$ ensures that

$$\nu^h(A) \leq \inf \left\{ \sum_{j=1}^{n} c_j h(\mathrm{diam}(B_j)) : c_j \geq 0, \chi_A \leq \sum c_j \chi_{B_j}, \mathrm{diam}(B_j) \leq t_k \right\}$$

so

$$\nu^h(A) \leq \sum_{j=1}^{n(k)} c_j^{(k)} h(t_j) \leq c < \infty. \quad \square$$

Lemmas 2.4 and 2.6 show us the way to the end, namely the proof Bandt's theorem, which will follow from the following.

LEMMA 2.7 (Principal Lemma). *For each $\epsilon > 0$ there is a $t_0 > 0$ so that if U is an open subset of G with diam$(U) \leq t_0$ then for some $s_1, \ldots, s_n \in G$ and $\alpha_1, \ldots, \alpha_n > 0$ we have*

$$\chi_A \leq \sum_{i=1}^{n} \alpha_i \chi_{s_i \cdot U}$$

and

$$\lambda(A) \leq \sum_{i=1}^{n} \alpha_i \lambda(s_i \cdot U) \leq (1+\epsilon)\lambda(A).$$

This proof depends on Cartan's Approximation Theorem (Theorem 1.8) and is rather delicate. We postpone the proof until after seeing what it buys us - the completion of the proof of Bandt's theorem. So principal lemma in hand, let's show how to put Lemmas 2.4 and 2.6 into play.

COROLLARY 2.8. *The fractional Hausdorff measure is non-trivial; in fact,*

$$0 < \nu^h(A) < \infty.$$

PROOF. If $\epsilon > 0$ then we can choose $t > 0$ so that $t < \min\{t_0, \epsilon\}$ and h is continuous at t. We can do this since h is monotone and so is continuous at all but countably many points of $(0, \min\{t_0, \epsilon\})$.

We can find an open set B in G with $\mathrm{diam}(B) \leq t$ so that

$$h(t) \leq (1+\epsilon)^2 \lambda(B).$$

How can we do this? Well if we pick $t' < t$ so that $h(t') \leq h(t)$, and since h is continuous at t,

$$h(t) < (1+\epsilon)h(t'),$$

then we choose C so that $\mathrm{diam}(C) \leq t'$ and $h(t') \leq (1+\epsilon)\lambda(C)$. Then

$$B = \left\{ x \in G : \rho(x, C) < \frac{t - t'}{2} \right\}$$

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will do. By our Principal Lemma we have a functional covering of A: there is $g_1, \dots, g_n \in G$ and $\alpha_1, \dots, \alpha_n > 0$ so that

$$\chi_A \leq \sum_{i=1}^n \alpha_i \chi_{g_i \cdot B}$$

with

$$\lambda(B)(\sum_{i=1}^n \alpha_i) \leq \sum_{i=1}^n \alpha_i \lambda(\alpha_i \cdot B) \leq (1+\epsilon)\lambda(A).$$

It follows that

$$\begin{aligned} h(t)E(t) &= h(t)E(t) \leq h(t) \sum_{i=1}^n \alpha_i \\ &\leq (1+\epsilon)^2 \lambda(U) \sum_{i=1}^n \alpha_i \\ &\leq (1+\epsilon)^3 \lambda(A). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary,

$$\liminf_{t \searrow 0} h(t)E(t) \leq \lambda(A) < \infty,$$

and so by Lemma 2.6, $\nu^h(A) < \infty$. We already know that $0 < \nu^h(A)$ and so by Note 2.3, ν^h is $\rho-$invariant with $0 < \nu^h(A) < \infty$. □

Proof. (of Principal Lemma) Since λ is regular and A is compact we can find an open set V such that

$$V \subseteq \{x \in G : \rho(x, A) < b\}$$

that contains A, has $\overline{V}$ compact and satisfies

$$\lambda(V) \leq (1+\epsilon)^{1/3}\lambda(A).$$

To see this, for each $a \in A$, let U_a be an open set containing a such that

$$U_a \subseteq \overline{U}_a \subseteq \{x : \rho(x, A) < b\}.$$

Let $a_1, \dots, a_n \in A$ so that

$$A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}.$$

Then

$$V = U_{a_1} \cap \cdots \cap U_{a_n}$$

is open, contains A, $\overline{V}$ is compact and $\overline{V} \subseteq \{x : \rho(x, A) < b\}$.

Let

$$W = \{x \in U : \rho(x, A) < b/2\}.$$

Let $f : G \rightarrow [0, 1]$ be a continuous function that is one on A and vanishes outside of W. Choose $\alpha > 0$ so that $\alpha[1 + (1+\epsilon)^{1/3}] < (1+\epsilon)^{1/3} - 1$, that is, $\alpha < 1 - (1+\epsilon)^{-1/3}$. Then

$$\frac{1+\alpha}{1-\alpha} < (1+\epsilon)^{1/3}.$$

We appeal to Cartan's Approximation Scheme to get an open set U_0 that contains G's identity e for which if $\phi \in \mathcal{K}^+(G)$ and U_0 contains the support of ϕ then for some $g_1, \dots, g_n \in \operatorname{supp}(f)$ and some $c_1, \dots c_n \geq \alpha$ we have

$$\left| f(g) - \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g) \right| \leq \alpha$$

for all $g \in G$. Let $t_0 < \min\{b/2, d(e, U_0^c)\}$ be a positive number. We'll show that this is the t_0 claimed in the Principal Lemma.

Let B be an open set with $\operatorname{diam}(B) \leq t_0$. Notice that if $g \in B$ then $g^{-1}B$ is an open set that contains e and $\operatorname{diam}(g^{-1}B) = \operatorname{diam}(B) \leq t_0$, since d is left invariant. Each point of B is within t_0 of g and so each point of $g^{-1}B$ is within t_0 of e. Can any $x \in g^{-1}B$ also be in U_0^c? If we try to imagine such an x then

$$d(e, x) \leq t_0 < d(e, U_0^c),$$

an impossibility. So (replacing B with $g^{-1}B$ if necessary), we can assume our B in the opening line of this paragraph contains e and is open with $\operatorname{diam}(B) \leq t_0$ and $B \subseteq U_0$.

Now λ is inner regular so we can choose a compact $C \subseteq B$ so $\lambda(C)$ is almost $\lambda(B)$, that is,

$$\lambda(B) \leq (1 + \epsilon)^{1/3} \lambda(C).$$

Suppose $\phi : G \to [0, 1]$ is a continuous function for which

$$\chi_C \leq \phi \leq \chi_B \leq \chi_{U_0}.$$

We know that ϕ is a member of $\mathcal{K}^+(G)$ with $\operatorname{supp}(\phi) \subseteq U_0$ and so Cartan's scheme tells us we can find $g_1, \dots, g_n \in \operatorname{supp}(f)$ and $c_1, \dots, c_n \geq 0$ so that for any $g \in G$

$$\left| f(g) - \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g) \right| \leq \alpha;$$

alternatively,

$$f(g) - \alpha \leq \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g) \leq f(g) + \alpha$$

for any $g \in G$. Since $\chi_A \leq f$,

$$\chi_A(g) - \alpha \leq \sum_{i=1}^{n} c_i \, {}_{g_i}\phi(g)$$

for any $g \in G$. Let $\phi_i = {}_{g_i}\phi$ and $d_i = \frac{c_i}{1-\alpha}$. Plainly $\phi_i \leq \chi_{g_i B}$ (since $\phi \leq \chi_B$) and

$$\chi_A \leq \sum_{i=1}^{n} d_i \phi_i \leq \sum_{i=1}^{n} d_i \chi_{g_i B}.$$

Now $\sum_{i=1}^{n} d_i \chi_{g_i B}$ is zero in V^c; after all, $g_1, \dots, g_n$ are in the support of f and

$$d(W, V^c) > b/2 \geq \operatorname{diam}(B)$$

so

$$\sum_{i=1}^{n} d_i \phi_i \leq (1 + \epsilon)^{1/3} \chi_V.$$

It follows that

$$\begin{aligned} \sum_{i=1}^{n} d_i \int \phi \, d\lambda &= \sum_{i=1}^{n} d_i \int \phi_i \, d\lambda = \int \sum_{i=1}^{n} d_i \phi_i \, d\lambda \\ &\leq (1+\epsilon)^{1/3} \int \chi_V \, d\lambda = (1+\epsilon)^{1/3} \lambda(V). \end{aligned}$$

Consequently

$$\begin{aligned} \sum_{i=1}^{n} d_i \lambda(g_i B) &= \lambda(B) \sum_{i=1}^{n} d_i \leq (1+\epsilon)^{1/3} \lambda(C) \sum_{i=1}^{n} d_i \\ &\leq (1+\epsilon)^{1/3} \int \phi \, d\lambda \sum_{i=1}^{n} d_i \leq (1+\epsilon)^{2/3} \lambda(V) \leq (1+\epsilon)\lambda(A) \end{aligned}$$

and that's that. □

Proof. (of Bandt's Theorem - Theorem 2.2) By Corollary 2.8 ν^h is a non-trivial left invariant Haar measure on G, so it is a multiple of λ. Since sets in G which are $\rho-$isometric have the same ν^h values, it follows that these sets have the same left Haar measure. □

1. Notes and Remarks

Whenever G is a locally compact metrizable topological group, G has a base for its topology consisting of open sets with compact closure; the collection of open balls with respect to the metric generates G's topology also forms a base for its topology. When can one find a left-invariant metric generating G's topology all of whose balls have compact closure? Of course, for such a thing to be so, G must be separable: after all, G is the union of the $n-$balls centered at the identity; if each of these have compact closure then it's easy to see that G has a countable dense subset - the union of the countable dense subsets of each $n-$ball will do.

Our next result, also due to R.A. Struble, tells us that this is the whole story. Though the topic of this result of Struble is not germane to the study of Haar measure, it is simply too satisfying a result not to be included.

Here's the theorem of R. Struble that led him to consider Theorem 2.1.

Theorem 2.9. *A locally compact group metrizable topological group has a left invariant metric that generates its topology in which all its open balls have compact closure if and only if G satisfies the second axiom of countability.*

Lemma 2.10. *Let G be a locally compact, second countable (hence metrizable, separable) group. Then there exists a family $\{U_r : r > 0\}$ such that*

(i) for each r, each U_r is open and $\overline{U}_r$ is compact,
(ii) $U_r = U_r^{-1}$
(iii) $U_r U_s \subseteq U_{r+s}$ (so if $r < s$ then $U_r \subseteq U_r U_{r-s} \subseteq U_s$),
(iv) $\{U_r : r > 0\}$ is a base for the open sets about e,
(v) $\cup_{r>0} U_r = G$.

Once Lemma 2.10 is established, we're ready for business. Indeed, let $\{U_r : r > 0\}$ be the family of open sets about e generated from Lemma 2.10. For $x, y \in G$, set

$$d(x, y) = \inf\{r : y^{-1}x \in U_r\}.$$

- Since $G = \cup_{r>0} U_r$, for an pair $x, y \in G$, we have $y^{-1}x \in U_r$ for some $r > 0$. It follows that $d(x, y) \geq 0$.
- $e \in U_r$ for each $r > 0$ so $d(x, x) = 0$.
- If $y^{-1}x \neq e$ then there is an $r_0 > 0$ so that $y^{-1}x \notin U_{r_0}$ (Part (iv) of Lemma 2.10 tells us this). But whenever $0 < r_0 < r$ we have (by Part (iii) of Lemma 2.10)
$$U_{r_0} \subseteq U_{r_0} U_{r-r_0} \subseteq U_r,$$
so $d(x, y) \geq r_0 > 0$.
- $U_r = U_r^{-1}$ so $y^{-1}x \in U_r$ precisely when $x^{-1}y \in U_r$; consequently, $d(x, y) = d(y, x)$.
- Suppose $x, y, z \in G$ with $y^{-1}x \in U_r, z^{-1}y \in U_s$. Then
$$z^{-1}x = z^{-1}yy^{-1}x \in U_s U_r \subseteq U_{r+s},$$
so $d(x, z) \leq r + s$. This is so whenever $y^{-1}x \in U_r$ so $d(x, z) \leq d(x, y) + s$; again this is so whenever $z^{-1}y \in U_s$ so $d(x, z) \leq d(x, y) + d(y, z)$.
- Finally, if $x, y, z \in G$ then
$$d(zx, zy) = \inf\{r : (zy)^{-1}zx \in U_r\} = \inf\{r : y^{-1}x \in U_r\} = d(x, y).$$

To summarize: d is a left invariant metric on G.

Since $d(x, e) < r$ means $x = e^{-1}x \in U_r$, the open d-ball of radius r centered at e is contained in U_r. Also this same d-ball contains $U_{r'}$ for any $0 < r' < r$ since if $x \in U_{r'}$ then

$$e^{-1}x = x \in U_{r'} \subseteq U_{r'} U_{\frac{r-r'}{2}} \subseteq U_{\frac{r+r'}{2}},$$

and so $d(x, e) \leq \frac{r+r'}{2} < r$. Therefore if $0 < r' < r$ then

$$U_{r'} \subseteq \{x : d(x, e) < r\} \subseteq U_r.$$

It follows that the open $d-$balls of radius r about e are cofinal with the collection $\{U_r : r > 0\}$ so the closure of each open $d-$ball is compact and d generates G's topology.

Proof. (Lemma 2.10) Let ρ be the left invariant metric resulting from Theorem 2.1. We can assume that each of the open balls

$$B_r = \{x \in G : \rho(x, e), r\}$$

has compact closure for $0 < r \leq 2$; after all, there is an r_0 so that for $r < r_0$, $\overline{B}_{r_0}$ is compact by G's locally compact nature so recalibrate ρ to make $r_0 = 2$ if necessary.

For $0 < r < 2$ we let $U_r = B_r$. This assures us clearly of (iv) and since we'll keep these U_r's, (iv) is assumed henceforth. Also (i), (ii), and (iii) hold when $r + s < 2$ by ρ's left invariant metric nature.

G is locally compact and satisfies the second countability axiom so G admits a countable open base

$$\{W_{2^n} : n \in \mathbb{N}\}$$

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for its topology, where we can (and do) assume that $\overline{W}_{2^n}$ is compact for each n. We define

$$U_2 = B_2 \cap W_2.$$

It's easy to verify that (i) and (ii) hold for $0 < r < 2$ and if $r + s < 2$ then (iii) holds as well.

We'll now inch our way from from (i), (ii), and (iii), $(r + s \leq 2)$ holding for $0 < r \leq 2$ to $0 \leq r \leq 4$.

First we have to define U_r for $2 < r < 2^2$. Let $0 < r < 2^2$. Set

$$U_r = \bigcup U_{t_1} \cdots U_{t_m}$$

where the union extends over all $t_1, \dots t_m$ so that each t_i satisfies $0 < t_i \leq 2$ and $t_1 + \cdots + t_m = r$.

If $2 < r < 2^2$ and $t_1 + \cdots + t_m = r$ where each $t_i > 0$ then there must be $k, l \in \mathbb{N}$ so that $1 \leq k < l < m$ and $t_1 + \cdots + t_k \leq 2$, $t_{k+1} + \cdots + t_l \leq 2$, and $t_{l+1} + \cdots + t_m \leq 2$. Why is this so? Well let k be the least j_1 so that $t_1 + \cdots + t_{j_1} \leq 2$, and let l be the least j_2 so that $t_{j_1+1} + \cdots + t_{j_2} \leq 2$. Then $\sum_{j_2+1}^{m} t_j \leq 2$ because otherwise, $t_{j_k+1} + \cdots + t_m > 2$ and $t_1 + \cdots + t_{j_1+1} \geq 2$ too where $j_1 + 1 < j_2 + 1$.

It follows that

$$\begin{aligned} U_{t_1} \cdot U_{t_2} \cdots U_{t_m} &\subseteq (U_{t_1} \cdots U_{t_k})(U_{t_{k+1}} \cdots U_{t_l})(U_{t_{l+1}} \cdots U_{t_m}) \\ &\subseteq U_{t_1 + \cdots + t_k} U_{t_{k+1} + \cdots + t_l} U_{t_{l+1} + \cdots + t_m} \quad \text{by (iv)} \\ &\subseteq U_2 \cdot U_2 \cdot U_2, \end{aligned}$$

so $U_r \subseteq U_2 \cdot U_2 \cdot U_2$ whenever $0 < r < 2^2$. Now $\overline{U}_2$ is compact so $\overline{U_2 \cdot U_2 \cdot U_2} \subseteq \overline{U}_2 \cdot \overline{U}_2 \cdot \overline{U}_2$ is too and $\overline{U}_r$ is compact for $0 < r < 2^2$. Since

$$(U_{t_1} \cdots U_{t_m})^{-1} = U_{t_m}^{-1} \cdots U_{t_1}^{-1} = U_{t_m} \cdots U_{t_1},$$

we see that

$$\begin{aligned} U_r^{-1} &= \left(\bigcup U_1 \cdots U_{t_m}\right)^{-1} \\ &= \bigcup (U_1 \cdots U_{t_m})^{-1} \\ &= \bigcup U_{t_m}^{-1} \cdots U_1^{-1} \\ &= \bigcup U_{t_m} \cdots U_1 = U_r \end{aligned}$$

for $2 < r < 2^2$.

Finally if $r > 0, s > 0$ and $r + s < 2^2$ then on supposing $t_1, \dots, t_m, \tau_1, \dots, \tau_j$ are positive and satisfy $t_1 + \cdots + t_m = r, \tau_1 + \cdots + \tau_j = s$ then

$$(U_{t_1} \cdots U_{t_m})(U_{\tau_1} \cdots U_{\tau_j}) = U_{t_1} \cdots U_{t_m} U_{\tau_1} \cdots U_{\tau_j}$$

so

$$U_r \cdot U_s \subseteq U_{r+s}.$$

What if $r, s > 0$ and $r + s = 2^2$? Suppose $r = s = 2$. Then $U_r U_s = U_2 U_2$. If $2 < r$ then $s < 2$. But $r < 2^2$ so

$$U_r \subseteq U_2 U_2 U_2$$

and so

$$U_r U_s \subseteq (U_2 U_2 U_2) U_2.$$

Either way define

$$U_{2^2} = (U_2 U_2 \cdot U_2 U_2) \cup W_{2^2}.$$

Here's what is so for $0 < r < 2^2$:

- Each U_r is open and $\overline{U}_2$ is compact;
- $U_r^{-1} = U_r$;
- if $r, s > 0$ and $r + s \leq 2^2$ then $U_r U_s \subseteq U_{r+s}$.

We still have $\{U_r : 0 \leq r \leq 2^2\}$ as a basis for the topology of G about e and of course, $W_{2^2} \subseteq U_{2^2}$.

We continue from U_{2^2} to U_{2^3} in a similar fashion and inch our way forward in a straightforward modification of the above procedure.

The fact that at each stage we ensure $W_{2^n} \subseteq U_{2^n}$ allows us to conclude that

$$G = \bigcup_{n \in \mathbb{N}} W_{2^n} \subseteq \bigcup_{n \in \mathbb{N}} U_{2^n} \subseteq G$$

which is (v) of Lemma 2.10. □

THEOREM 2.11 (Braconnier). *If G is a locally compact topological group and G admits a bi-invariant metric d that determines its topology then G is unimodular.*

PROOF. Let λ be left Haar measure on G and assume that G is not unimodular. Let U be an open set containing e such that $\lambda(U) < \infty$. Let B be an open ball centered at e (of radius ρ) and so that $\overline{B}$ is compact. Then for any $x \in G$, the set xBx^{-1} is just B, thanks to d's bi-invariance, so

$$xBx^{-1} \subseteq U.$$

Let $x_0 \in G$ satisfy

$$\Delta(x_0^{-1})\lambda(B) > \lambda(U),$$

where Δ is the modular function of G. Then $x_0 B x_0^{-1} \subseteq U$ and

$$\lambda(x_0 B x_0^{-1}) = \Delta(x_0^{-1})\lambda(x_0 B) = \Delta(x_0^{-1})\lambda(B) > \lambda(U).$$

Oops! □

EXAMPLE 2.12. *The general linear group, $\mathcal{GL}(n;\mathbb{R})$, on $n-$space, $n \geq 2$, is unimodular. For each m, let*

$$X_m = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ 0 & m \end{pmatrix}, \quad \textit{and } Y_m = \begin{pmatrix} m & \frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix} \in \mathcal{GL}(n;\mathbb{R}).$$

Then

$$X_m Y_m = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ 0 & m \end{pmatrix} \begin{pmatrix} m & \frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{m^2} \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But

$$Y_m X_m = \begin{pmatrix} m & \frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix} \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ 0 & m \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Therefore no bi-invariant metric can be found so that it generates the topology of $\mathcal{GL}(n;\mathbb{R})$.

THEOREM 2.13. *If G is a locally compact metrizable topological group and ρ is a left invariant metric that generates G's topology then (G,ρ) is a complete metric.*

Indeed if U is an open set with compact closure and if $e \in U$ then there is an open ball B centered at e with $\overline{B}$ both compact and contained in U. Suppose R is the radius of B and let (g_n) be a $\rho-$Cauchy sequence in (G, ρ). Then there is an $N \in \mathbb{N}$ so for $m, n \leq N$,

$$\rho(g_n, g_m) < \frac{R}{3};$$

it soon follows that for $n \geq N$,

$$\rho(g_n, g_N) < \frac{R}{3}.$$

Therefore for $n \geq N$, g_n lies in the compact, closed ball of radius $\frac{R}{3}$ and so (g_n) must converge.

Observation 2.14. *Let U be an open set in the topological group G with $e \in G$ and suppose that K is a compact subset of G. Then there is an open set V in G with $e \in V$ so that*

$$xVx^{-1} \subseteq U$$

for every $x \in K$.

Proof. To see this, let $\mathcal{W}$ denote the collection of all open sets W in G such that $W = W^{-1}$. We claim that for any $y \in G$ there is a $V \in \mathcal{W}$ so that if $x \in Vy$ then

$$xVx^{-1} \subseteq U.$$

In fact, we can pick $V_1 \in \mathcal{W}$ so that $V_1 \cdot V_1 \cdot V_1 \subseteq U$ and we can pick $V_2 \in \mathcal{W}$ so that

$$yV_2y^{-1} \subseteq V_1.$$

(This is thanks to the continuity of $x \to ax \to axa^{-1}$ for any $a \in G$.) Let

$$V = V_1 \cap V_2.$$

Then if $x \in Vy$,

$$xy^{-1} \in V \subseteq V_1,$$

and

$$yx^{-1} = (x^{-1}y)^{-1} \in V^{-1} = V \subseteq V_1,$$

and so

$$xVx^{-1} \subseteq xV_2x^1 = (xy^{-1})\left(yV_2y^{-1})\right)(yx^{-1}) \subseteq V_1 \cdot V_1 \cdot V_1 \subseteq U.$$

So for each $y \in K$ there is a $V_y \in \mathcal{W}$ so that $x \in V_xy$ implies $xV_yx^{-1} \subseteq U$. But

$$K \subseteq \bigcup_{y \in K} V_yY,$$

and each V_yy is open; hence we can find $y_1, \ldots, y_n \in K$ so that

$$K \subseteq (V_{y_1}y_1) \cup \cdots \cup (V_{y_n}y_n).$$

Let

$$V = V_{y_1} \cap \cdots \cap V_{y_n}.$$

Then if $x \in K$, it must be that $x \in V_{y_k}y_k$ for some $k, 1 \leq k \leq n$, and so

$$xVx^{-1} \subseteq xV_{y_k}y_k \subseteq U.$$

□

Theorem 2.15. *A compact metrizable topological group G admits a bi-invariant metric that generates its topology.*

PROOF. Let ρ be a left invariant metric on G that generates G's topology. For $x, y \in G$ define

$$d(x,y) = \sup\{\rho(xz, yz) : z \in G\}.$$

Then d is finite for all $x, y \in G$ and is easily seen to be bi-invariant.

Suppose $\epsilon > 0$. By our observation above, there is a $\delta > 0$ so that

$$z^{-1} \cdot \{x \in G : \rho(x,e) < \delta\} \cdot z \subseteq \{x \in G : \rho(x,e) < \epsilon\},$$

for all z. It follows that $\rho(x,e) < \delta$ ensures that $d(z^{-1}xz, e) = d(xz, z) < \epsilon$ for all z and so $d(x,e) \leq \epsilon$. The open $\rho-$ball of radius δ centered at e is contained in the closed $d-$ball of radius ϵ centered at e.

It is plain that the open $d-$ball of radius ϵ centered at e is contained in the open $\rho-$ball of radius ϵ centered at e. Therefore ρ and d generate the same topology. □

Moreover

- *If G is a topological group of the second category and H is a subgroup of G then $G\backslash H$ is either empty or of the second category in G.*

  Let $y \in G\backslash H$. Then $yH \in G\backslash H$ (distinct cosets are disjoint). Therefore should $G\backslash H$ be of the first category then so is yH and from this we conclude that $G = yH \cup (G\backslash H)$ is of the first category.
- *If G is a topological group of the second category and H is a dense $\mathcal{G}_\delta$ subgroup of G then $H = G$.*

  After all, $H = \cap_n H_n$ where each H_n is a dense open subset of G and so $G\backslash H_n$ is closed and nowhere dense for each n; it follows that $G\backslash H = \cup_n (G\backslash H_n)$ is of the first category, and so by the previous remark, $G\backslash H = \emptyset$.

THEOREM 2.16 (V.Klee). *Let G be a topological group with a bi-invariant metric ρ which generates G's topology. Suppose (G, ρ) admits a complete metric d that generates G's topology. Then G is actually complete under ρ.*

PROOF. Let (G^*, ρ^*) be the completion of (G, ρ) Then (G^*, ρ^*) is a topological group into which (G, ρ) is naturally isomorphically and isometrically embedded as a dense subgroup. But topological complete metric spaces are always $\mathcal{G}_\delta-$sets in any super metric space, thanks to an oldie but goodie of Sierpinski. Hence $G = G^*$ and so G is $\rho-$complete.