CHAPTER 1

Marriage and Haar Measure

Suppose for each boy b_i in the set B of n boys:

$$B = \{b_1, \cdots, b_n\}$$

there is a collection $G(b_i)$ of girls with whom b_i is acquainted.

Theorem 1.1. *With bigamy an anathema, for each boy in B to be able to marry a girl with whom he is acquainted it is both necessary and sufficient that regardless of $C \subseteq B$*

$$\left| \bigcup_{b \in C} G(b) \right| \geq |C| \tag{0.1}$$

holds.

Since it is plain that (0.1) is necessary, we'll concentrate on proving sufficiency of (0.1). So we suppose (0.1) to be in effect and prove the possibility of a wise matchmaker by an induction on the number of boys in B. Though not universally associated with marriage, we nevertheless introduce the notion of perfect cliques.

Definition 1.2. *A clique $C \subseteq B$ is* perfect *if*

$$\left| \bigcup_{b \in C} G(b) \right| = |C| .$$

Lemma 1.3. *If C and C' are perfect then so is $C \cup C'$.*

Proof. By (0.1) we know that

$$\left| \bigcup_{b \in C \cup C'} G(b) \right| \geq \left| C \cup C' \right| .$$

Further

$$|C \cup C'| = |C| + |C'| - |C \cap C'|,$$

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and since it's too much to expect all the $G(b)$'s to be singletons

$$\begin{aligned}\left|\bigcup_{b\in C} G(b)\right| &\leq \left|\bigcup_{b\in C} G(b)\right| + \left|\bigcup_{b\in C'} G(b)\right| - \left|\bigcup_{b\in C\cap C'} G(b)\right| \\ &= |C| + |C^{'}| - \left|\bigcup_{b\in C\cap C'} G(b)\right| \\ &\leq |C| + |C^{'}| - |C\cap C^{'}| \text{ by } (0.1) \\ &= |C\cup C^{'}|.\end{aligned}$$

So $C\cup C^{'}$ is perfect. □

So for now we suppose that for any collection of fewer than n boys, (0.1) is sufficient to warm the cockles of the matchmaker's cold heart. Let B be a collection of n boys which given any $C\subseteq B$ we have

$$\left|\bigcup_{b\in C} G(b)\right| \geq |C|.$$

We'll take a good luck at $G(b_n)$.

If $g\in G(B_n)$ then looking at

$$G(b_1)\setminus\{g\}, G(b_2)\setminus\{g\}, \ldots, G(b_{n-1})\setminus\{g\}$$

we see two possibilities:

MAYBE for some $g\in G(b_n)$

$$G(b_1)\setminus\{g\}, G(b_2)\setminus\{g\}, \ldots, G(b_{n-1})\setminus\{g\}$$

satisfies (0.1)and so we can effect a marriage of each $b_1, b_2, \ldots, b_{n-1}$ to a girl of his acquaintances and still have the lovely g left to marry b_n when $g\in G(b_n)$. Of course in this case we're happy onlookers and the proof is done.

MAYBE for each $g\in G(b_n)$, the collection

$$G(b_1)\setminus\{g\}, G(b_2)\setminus\{g\}, \ldots, G(b_{n-1})\setminus\{g\}$$

fails (0.1). i.e., for each $g\in G(b_n)$ there is a $C_g\subseteq\{b_1, b_2, \ldots, b_{n-1}\}$ such that

$$\left|\bigcup_{b_i\in C_g} G(b_i)\setminus\{g\}\right| < |C_g|.$$

But deleting g from $G(b_i)$ can change the number of girls in this collection by at most 1 and

$$\{G(b_i) : b_i\in C_g\}$$

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satisfies (0.1) by our inductive hypothesis so each of the $G(b_i)$'s, with $b_i \in C_g$ must contain g! It follows that

$$\left| \bigcup_{b_i \in C_g} G(b_i) \setminus \{g\} \right| = \left| \bigcup_{b_i \in C_g} G(b_i) \right| - 1 < |C_g|.$$

Since $\{G(b_i) : b_i \in C_g\}$ satisfies (0.1) it must be that

$$\left| \bigcup_{b_i \in C_g} G(b_i) \right| = |C_g|.$$

i.e., $\{G(b_i) : b_i \in C_g\}$ is perfect for each $g \in G(b_n)$.

Our lemma assures now that $\{G(b_i) : b_i \in \cup_{g \in G(b_n)} C_g\}$ is also perfect and so

$$\left| \bigcup \{G(b_i) : b_i \in \cup_{g \in G(b_n)} C_g\} \right| = \left| \bigcup_{g \in G(b_n)} C_g \right|.$$

Keep in mind that

$$G(b_n) \subseteq \bigcup_{b_i \in C_g} G(b_i).$$

So if we add b_n to $\cup\{C_g : g \in G(b_n)\}$ we have

$$\begin{aligned} \left| G(b_n) \bigcup \{G(b_i) : b_i \in \cup_{g \in G(b_n)} C_g\} \right| &= \left| \bigcup \{G(b_i) : b_i \in \cup_{g \in G(b_n)} C_g\} \right| \\ &= \left| \bigcup \{C_g : g \in G(b_n)\} \right| \\ &< \left| \{b_n\} \bigcup \left\{ \bigcup_{g \in G(b_n)} C_g \right\} \right|, \end{aligned}$$

and we've found a collection $C \subseteq B$,

$$C = \{b_n\} \bigcup \left\{ \bigcup_{g \in G(b_n)} C_g \right\}$$

which violates (0.1).

We see then that the second of the possible outcomes is defined to fail and so we deduce that the first 'MAYBE' is. □

We will apply our result to proving the existence of a probability measure μ on a compact metric space S that is $G-$invariant, where G is the group of isometries of S onto itself. The use of the solution to the marriage problem in this context is due to Maak.

Let S be a compact metric space and G be the group of isometries of S onto S. Let $\epsilon > 0$. Let N_ϵ be a minimal $\epsilon-$net. i.e,

$$S = \bigcup_{k \in N_\epsilon} \{y \in S : d(x, y) < \epsilon\},$$

and $|N_\epsilon|$ is minimal.

For $f \in C(S)$, define

$$\mu_{N_\epsilon}(f) = \frac{1}{|N_\epsilon|} \sum_{x \in N_\epsilon} f(x).$$

Clearly μ_{N_ϵ} is a probability measure on S and as such belongs to $B_{C(S)^*}$, the unit ball of $C(S)^*$. Since $C(S)$ is a separable Banach space, $B_{C(S)^*}$ is compact and metrizable in the weak* topology. Hence for a suitably chosen sequence $\epsilon_n \searrow 0$, we have that $(\mu_{N_{\epsilon_n}})$ is weak* convergent to a probability $\mu \in B_{C(S)^*}$.

Our first task will be to show that μ is independent of the choices of N_ϵ made, $\epsilon-$by-ϵ. To this end we will show that if $\widetilde{N_\epsilon}$ is another minimal $\epsilon-$net then not only are N_ϵ and $\widetilde{N_\epsilon}$ equinumerous but we can find a bijection ϕ of N_ϵ onto $\widetilde{N_\epsilon}$ such that for any $x \in N_\epsilon$,

$$d(x, \phi(x)) < 2\epsilon.$$

We'll say that $x \in N_\epsilon$ and $y \in \widetilde{N_\epsilon}$ are *acquainted* if

$$\{s \in S : d(x,s) < \epsilon\} \cap \{s \in S : d(y,s) < \epsilon\} \neq \emptyset.$$

We claim that if K is any subset of N_ϵ and L is the subset of $\widetilde{N_\epsilon}$ consisting of s's that are acquainted with some member of K then L has at least as many members as K does. Why is this so? Well, L consists of precisely those s's in $\widetilde{N_\epsilon}$ such that

$$\{s^{'} \in S : d(s, s^{'}) < \epsilon\} \cap (\cup_{t \in N_\epsilon}\{s^{'} \in S ; d(t, s^{'}) < \epsilon\}) \neq \emptyset.$$

Were L to have fewer members than K then the set

$$L \cup (N_\epsilon \setminus K)$$

would have cardinality

$$\begin{aligned} |L \cup (N_\epsilon \setminus K)| &\leq |L| + |(N_\epsilon \setminus K)| \\ &< |K| + |(N_\epsilon \setminus K)| = |N_\epsilon|. \end{aligned}$$

Yet for any $x \in S$, x within ϵ of some point of N_ϵ, maybe x is within ϵ of a point of L and maybe not! But if x is *not* within ϵ of a point of L, it is *not* within ϵ of a point of K – after all, that's how L was defined! So x is within ϵ of a point $(N_\epsilon \setminus K)$. Then $L \cup (N_\epsilon \setminus K)$ is an $\epsilon-$net for S with fewer members than N_ϵ. Oops! Our claim is verified.

Now our marriage counselor steps in to advise us that there is a bijection ϕ of N_ϵ onto $\widetilde{N_\epsilon}$ such that for each $t \in N_\epsilon$, $\phi(t)$ is acquainted with t. i.e., there is a bijection $\phi : N_\epsilon \to \widetilde{N_\epsilon}$ such that for each $t \in N_\epsilon$,

$$d(t, \phi(t)) < 2\epsilon.$$

But this entails the following: if $f \in C(S)$ then

$$\begin{aligned} |\mu_{N_\epsilon}(f) - \mu_{\widetilde{N_\epsilon}}(f)| &\leq \frac{1}{|N_\epsilon|} \sum_{t \in N_\epsilon} |f(t) - f(\phi(t))| \\ &\leq \omega_f(2\epsilon), \end{aligned}$$

where

$$\omega_f(\eta) := \sup\{|f(t) - f(s)| : d(s,t) \leq \eta\}$$

is the modulus of (uniform) continuity of $f \in C(S)$. It follows from this that

$$\text{weak} * \lim_n \mu_{N_{\epsilon_n}}, \text{weak} * \lim_n \mu_{\widetilde{N_{\epsilon_n}}},$$

coexist and coincide.

What of it? Well if $g \in G$ then

$$\widetilde{N_\epsilon} = \{g(t) : t \in N_\epsilon\}$$

is a minimal $\epsilon-$net if N_ϵ is, so for any $f \in C(S)$

$$\begin{aligned} \mu(f \circ g) &= \lim_n \frac{1}{|N_\epsilon|} \sum_{t \in N_\epsilon} f(g(t)) \\ &= \lim_n \frac{1}{|N_\epsilon|} \sum_{s \in \widetilde{N_\epsilon}} f(s) \\ &= \lim_n \mu_{\widetilde{N_{\epsilon_n}}}(f) = \mu(f), \end{aligned}$$

and μ is $G-$invariant. $\square$