# The fundamental duality formula in set-valued optimization

**Andreas H. Hamel**

Princeton University & Yeshiva University New York

ahamel@princeton.edu, hamel@yu.edu

## $\|\blacktriangleright$ Main result.

- $X, Y, Z$ separated locally convex spaces with top. duals $X^*, Y^*, Z^*$,
- $C \subseteq Z$ convex cone with $0 \in C$,
  $C^* = \{z^* \in Z^* \colon \forall z \in C \colon z^*(z) \leq 0\}$ (negative) dual cone of $C$,
- $F \colon X \times Y \to \mathcal{P}(Z)$ with $\mathcal{P}(Z)$ set of all subsets of $Z$ including $\emptyset$.

**Theorem.** (preliminary version) Under usual assumptions,

$$\inf_{x \in X} F(x, 0) = \max_{y^* \in Y^*,\, z^* \in C^* \setminus \{0\}} -F^*(0, y^*, z^*).$$

**Scalar version.** (Zalinescu 2002, Thm. 2.7.1) With $F \colon X \times Y \to \mathbb{R} \cup \{+\infty\}$, under usual assumptions

$$\inf_{x \in X} F(x, 0) = \max_{y^* \in Y^*} -F^*(0, y^*).$$

**Theorem.** (preliminary version) Under usual assumptions,

$$\inf_{x\in X} F(x,0) = \max_{y^*\in Y^*,\, z^*\in C^*\setminus\{0\}} -F^*(0,y^*,z^*).$$

**Questions.**

- Inf/sup for set-valued functions? Min/max?
- What about the additional dual variable $z^* \in C^*\setminus\{0\}$?
  Why not $S \in \mathcal{L}(X,Z)$, $T \in \mathcal{L}(Y,Z)$ instead of $x^*$, $y^*$?
- Relationships to "vector" results?
- Usual assumptions: Convexity, interior point conditions? Assumptions for ordering cone $C$?
- Usual proof: $\partial v(0) \neq \emptyset$ with $v(y) = \inf_{x\in X} F(x,y)$?
- Why bother at all? What about applications?

## ||▶ Rest of the talk.

**(0)** Motivation: Risk measures for markets with transaction costs.

**(1)** Image spaces and inf/sup for set-valued functions.

**(2)** Min/max: a solution concept in set-valued optimization.

**(3)** Dual variables: set-valued "conlinear" functions.

**(4)** Legendre-Fenchel conjugates for set-valued functions.

**(5)** Perturbation in set-valued optimization, FDF again.

## ‖ ▶ Motivation.

### Risk measures in markets with transaction costs.

- Idea of a risk measure: Find minimal deposit (in dollar) for the payoff $X : \Omega \to \mathbb{R}$ of a 'risky' business, thus

$$\varrho_A(X) = \inf\{t \in \mathbb{R} \colon X + t\mathbb{1} \in A\}$$

  where $A \subseteq L^0(\Omega, \mathcal{F}, P)$ is a set of acceptable positions.

- Idea of markets with transaction costs: There is a (proportional) difference between bid price (you sell) $p_b(X)$ and ask price (you buy) $p_a(X)$: $p_b(X) < p_a(X)$.

- Idea of multiasset models: asset vectors $X : \Omega \to \mathbb{R}^d$, $d > 1$, and every transaction of asset # i into asset #j is subject to a bid-ask price spread.

- Set of positions which can be exchanged into a non-negative position at time $t = 0$ is a closed convex cone $K_0$ with $\mathbb{R}^d \subseteq K_0$.

- Idea of set-valued risk measures: Instead of looking for the above infimum, collect

$$R_A(X) = \left\{u \in \mathbb{R}^d \colon X + u\mathbb{1} \in A\right\}$$

where $A \subseteq L^0_d(\Omega, \mathcal{F}, P)$ is a set of acceptable positions in $d$ assets (currencies, shares of stocks, options etc.).

- Duality result for set-valued closed convex risk measures:

$$R_A(X) = \bigcap_{(Q,w)\in\mathcal{W}} \left\{-\alpha(Q, w) + F^M_{(Q,w)}[-X]\right\}$$

with $\mathcal{W} \subseteq \mathcal{M}^P_{1,d} \times K^+_0$ (Hamel/Heyde 2010, Hamel/Heyde/Rudloff 2010).

- Precursor: Y.M. Kabanov, *Hedging and liquidation in currency markets under transaction costs*, Fin. & Stoch. 3, 237-248 (1999)

## $\|\blacktriangleright$ Order complete lattices of sets.

**Recall.**

- $(Z, \leq_C)$ linear space with $\leq_C$ generated by a convex cone $C \subseteq Z$ with $0 \in C$
- $\mathcal{P}(Z)$ set of all subsets of $Z$ including $\emptyset$, $Z$.

**The vector order:**

$$z_1 \leq_C z_2 \Leftrightarrow z_2 \in z_1 + C \Leftrightarrow z_1 \in z_2 - C.$$

**Two canonical extensions of $\leq_C$ to $\mathcal{P}(Z)$:**

$$A \preccurlyeq_C B \iff B \subseteq A + C$$
$$A \curlyeqprec_C B \iff A \subseteq B - C.$$

**In general, they are different!**

- Inf-extension of $\leq_C$ to $\mathcal{P}(Z)$: $A \preccurlyeq_C B \Leftrightarrow B \subseteq A + C$.

  Sup-extension of $\leq_C$ to $\mathcal{P}(Z)$: $A \curlyeqprec_C B \Leftrightarrow A \subseteq B - C$.

**Image spaces.** For convexity/concavity, define ($Z$ topological linear space):

$$Q_C^t := Q_C^t(Z) := \{A \subseteq Z \colon A = \operatorname{cl}\operatorname{co}(A + C)\}$$
$$Q_t^C := Q_t^C(Z) := \{A \subseteq Z \colon A = \operatorname{cl}\operatorname{co}(A - C)\}.$$

**Result.** (easy) $A \in Q_C^t \;\Leftrightarrow\; -A \in Q_t^C$ and

$$A, B \in Q_C^t\colon \quad A \preccurlyeq_C B \quad \Leftrightarrow \quad A \supseteq B$$
$$A, B \in Q_t^C\colon \quad A \curlyeqprec_C B \quad \Leftrightarrow \quad A \subseteq B$$

**From now:** Only $\left(Q_C^t, \supseteq\right)$.

**Theorem.** $\left(Q^t_C, \supseteq\right)$ is an **order complete lattice** with

$$\inf_{Q^t_C} \mathcal{A} = \operatorname{cl}\operatorname{co} \bigcup_{A\in\mathcal{A}} A, \quad \sup_{Q^t_C} \mathcal{A} = \bigcap_{A\in\mathcal{A}} A.$$

for $\mathcal{A} \subseteq Q^t_C$, and for $B \in Q^t_C$,

$$\inf_{Q^t_C} \{A + B \colon A \in \mathcal{A}\} = \operatorname{cl}\left(\inf_{Q^t_C} \mathcal{A} + B\right) = \left(\inf_{Q^t_C} \mathcal{A}\right) \oplus B.$$

**Remark.** This is true without further assumption to $C$.

**Remark.** Hahn-Banach-Kantorovich is only true if the image vector space is a conditional order complete vector lattice (l.u.b. property). This is very restrictive, whereas in $Q^t_C$ this property comes for free.

**Question.** Sum in $Q^t_C$, $Q^C_t$: $A \oplus B = \operatorname{cl}(A + B)$. Algebraic structure?

**Image spaces (of sets): algebraic structure.**

**Definition.** A **conlinear space** is a set $W$ with addition and multiplication with non-negative real numbers such that

- $(W, +)$ is a commutative semigroup with neutral element $\theta$;
- For $s, t \geq 0$, $w, w_1, w_2 \in W$ it holds

$$0w = \theta,\ 1w = w,\ t\,(sw) = (ts)\,w,\ s\,(w_1 + w_2) = sw_1 + sw_2.$$

**Result.** $\left(Q^t_C, \supseteq, \oplus\right)$, $\left(Q^C_t, \subseteq, \oplus\right)$ are ordered conlinear spaces.

**Another example.** $\mathbb{R} \cup \{\pm\infty\}$ with inf–addition or sup-addition:
$(+\infty) + (-\infty) = +\infty$ or $(+\infty) + (-\infty) = -\infty$.

**Summary so far.** For $f\colon X \to Q^t_C$,

$$\inf_{x \in X} f(x) = \operatorname{cl} \operatorname{co} \bigcup_{x \in X} f(x)$$

$$\sup_{x \in X} f(x) = \bigcap_{x \in X} f(x).$$

$\left(Q^t_C, \supseteq, \oplus, \cdot\right)$ is an ordered conlinear space and an order complete, residuated lattice.

## $\|\blacktriangleright$ Set-valued convex functions.

**Set-valued functions.** $X$, $Z$ nontrivial, separated locally convex: A function $f\colon X \to Q_C^t$ is called convex (closed, sublinear) iff

$$\operatorname{gr} f = \{(x, z) \in X \times Z \colon z \in f(x)\}$$

is convex (closed, a convex cone).

**Result.** (easy) $f$ is convex if and only if

$$f(tx_1 + (1-t)x_2) \supseteq tf(x_1) \oplus (1-t) f(x_2)$$

whenever $t \in (0, 1)$, $x_1, x_2 \in X$. If $f$ is convex (closed), then $f(x)$ is convex (closed) for all $x \in X$.

**Examples.**

**Set-valued indicator function.** For $M \subseteq X$, the function

$$I_M(x) = \operatorname{cl} C, \text{ if } x \in M, \quad I_M(x) = \emptyset, \text{ if } x \notin M$$

is convex (closed) if and only if $M$ is convex (closed).

**Conlinear functions.** $x^* \in X^*$, $z^* \in C^- \setminus \{0\}$. The function

$$x \mapsto S_{(x^*,z^*)}(x) = \{z \in Z \colon x^*(x) + z^*(z) \leq 0\}$$

- maps into $Q^t_{H(z^*)} \subseteq Q^t_C$,
- is positively homogeneous and additive with $S_{(x^*,z^*)}(0) = H(z^*) = \{z \in Z \colon z^*(z) \leq 0\}$.

**Fenchel–Legendre conjugates.** $f\colon X \to Q^t_C$

$$-f^*(x^*, z^*) := \inf_{Q^t_C} \left\{ f(x) + S_{(x^*,z^*)}(-x) \colon x \in X \right\}$$
$$= \mathrm{cl} \bigcup_{x \in X} \left[ f(x) + S_{(x^*,z^*)}(-x) \right].$$

Scalar: $-f^*(x^*) = \inf_{x \in X} [f(x) + x^*(-x)]$.

$$f^{**}(x) := \sup_{Q^t_C} \left\{ -f^*(x^*, z^*) + S_{(x^*,z^*)}(x) \colon x^* \in X^*,\ z^* \in C^- \backslash \{0\} \right\}$$
$$= \bigcap_{x^* \in X^*,\ z^* \in C^- \backslash \{0\}} \left[ -f^*(x^*, z^*) + S_{(x^*,z^*)}(x) \right].$$

Scalar: $f^{**}(x) = \sup_{x^* \in X^*} [x^*(x) - f^*(x^*)]$.

**Fenchel–Moreau theorem.**

$f\colon X \to \mathcal{Q}^t_C$ proper closed convex, or identically $\emptyset$ or $Z$ if and only if $f = f^{**}$.

**Remark.** Such theorems for functions $f\colon X \to (Z, \leq_C)$ are only true under very strict assumptions for $Z$, $C$.

**Fenchel–Moreau theorem.**

$f\colon X \to \mathcal{Q}^t_C$ proper closed convex, or identically $\emptyset$ or $Z$

if and only if $f = f^{**}$.

**Corollary.** (Kabanov, discrete time).

$$\Gamma = \left\{ v \in \mathbb{R}^d \colon V_T^{v,L} \succeq C \text{ for some } L \in \mathfrak{A} \right\} = \bigcap_{Z \in \mathfrak{P}_0} \left\{ v \in \mathbb{R}^d \colon \widehat{Z}_0 v \geq E \widehat{Z}_T C \right\} = D.$$

**Proof.** $\Gamma$ is the value of a set-valued proper closed sublinear risk measure at $-C$, and $D$ is the value of its biconjugate at $-C$. (Note: $C$ here corresponds to $x$ in the general theory. The additional dual variable $z^*$ is $\widehat{Z}_0$.)

## ‖► Duality in set-valued optimization.

**The problem.** For $f\colon X \to \mathcal{P}(Z)$, find

$$\inf_{Q_C^t} \{f(x) : x \in X\} = \operatorname{cl}\operatorname{co} \bigcup_{x \in X} f(x).$$

**Perturbation.** $Y$ another locally convex space, $F\colon X \times Y \to Q_C^t$ with $F(x,0) = \operatorname{cl}\operatorname{co}\,(f(x) + C)$. Define $v\colon Y \to Q_C^t$ by

$$v(y) = \inf_{Q_C^t} \{F(x,y) : x \in X\} = \operatorname{cl}\operatorname{co} \bigcup_{x \in X} F(x,y).$$

Then

$$v(0) = \inf_{Q_C^t} \{F(x,0) : x \in X\} = \operatorname{cl}\operatorname{co} \bigcup_{x \in X} F(x,0)$$

is also a $Q_C^t$-valued extension of the original problem.

**The dual problem.** Introducing

$$w\left(x^*\right) = \sup_{Q_C^t}\left\{-F^*\left(x^*, y^*, z^*\right) : y^* \in Y^*,\ z^* \in C^- \setminus \{0\}\right\}$$
$$= \bigcap_{y^* \in Y^*,\, z^* \in C^- \setminus \{0\}} -F^*\left(x^*, y^*, z^*\right),$$

the dual problem is to determine

$$\text{(D)} \qquad w\left(0\right) = \sup_{Q_C^t}\left\{-F^*\left(0, y^*, z^*\right) : y^* \in Y^*,\ z^* \in C^- \setminus \{0\}\right\}.$$

**Weak duality.** Always

$$Z \supseteq w\left(0\right) = v^{**}\left(0\right) \supseteq v\left(0\right) \supseteq \emptyset.$$

**Proof.** Immediate from the definition of the conjugates.

## Strong duality - dual maximum.

Let $z^* \in C^- \backslash \{0\}$ be given. An element $y^* \in Y^*$ is called a $z^*$-solution of (D) iff

$$w(0) \oplus H(z^*) = -F^*(0, y^*, z^*).$$

A set $\Delta \subseteq Y^* \times C^* \backslash \{0\}$ is called a solution of (D) iff

(i) $y^*$ is a $z^*$-solution of (D) whenever $(y^*, z^*) \in \Delta$,

(ii) $w(0) = \sup_{\mathcal{Q}^t_C} \{-F^*(0, y^*, z^*) : (y^*, z^*) \in \Delta\}$.

**Recall.** $H(z^*) = \{z \in Z \colon z^*(z) \leq 0\}$.

**Fundamental duality formula.** $F$ convex in $(x,y)$, there is $(x_0,z_0) \in X \times Z$ such that $F(x_0,0) \neq \emptyset$ and $(0,z_0) \in \operatorname{int} \operatorname{gr} F(x_0,\cdot)$. Then either $v(0) = w(0) = Z$ or

$$\inf_{x\in X} F(x,0) = \max_{y^*\in Y^*,\, z^*\in C^*\setminus\{0\}} -F^*(0,y^*,z^*)$$

where 'max' means that there is a solution $\Delta$ of (D).

**Proof.** Two possibilities (at least):

- directly, using a separation argument in $Y \times Z$
- indirectly, using separation in $Z$ to get $z^*$, scalarizing, applying scalar FDF.

**Scalarization.** Define family $\varphi_{f,z^*}\colon X \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\varphi_{f,z^*}(x) = \inf\left\{-z^*(z) : z \in f(x)\right\}, \; z^* \in C^- \setminus \{0\},$$

and perturbations $\Phi_{F,z^*}\colon X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\Phi_{F,z^*}(x,y) = \inf\left\{-z^*(z) : z \in F(x,y)\right\}, \; z^* \in C^- \setminus \{0\}.$$

**Lemma.** For $f\colon X \to \mathcal{Q}^t_C$,

$$\forall x \in X\colon f(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z\colon \varphi_{f,z^*}(x) \leq -z^*(z)\right\},$$

and similar for $F$. Moreover,

$$-f^*(x^*, z^*) = \left\{z \in Z\colon -\varphi^*_{f,z^*}(x^*) \leq -z^*(z)\right\},$$

and similar for $-F^*$.

**Principal idea of the proof.** If $z^* \in C^* \setminus \{0\}$ is such that $v(0) \oplus H(z^*) \neq Z$, then $\Phi_{F,z^*}$ satisfies the assumptions of the scalar FDF, thus

$$\inf_{x \in X} \Phi_{F,z^*}(x,0) = \max_{y^* \in Y^*} -\Phi^*_{F,z^*}(0,y^*).$$

Using the lemma we get the result.

|| ▶ **Example: A picture of super-/subhedging prices.**

## ‖▶ Some references.

*A Duality Theory for Set-Valued Functions II: Duality Theorems*, almost finished, 2010

*Price Bounds for Markets with Transaction Costs*, with B. Rudloff, submitted, 2010

*Set-Valued Risk Measures for Conical Market Models*, with F. Heyde and B. Rudloff, submitted, 2010

*A Fenchel–Rockafellar Duality Theorem for Set-Valued Optimization*, submitted, 2009

*A Duality Theory for Set-Valued Functions I: Fenchel Conjugation Theory*, Set–Valued and Variational Analysis 17, 153–182, 2009;

*Duality for Set-Valued Measures of Risk*, with F. Heyde, SIAM J. on Financial Mathematics, 1(1), 66–95, 2010

PhD thesis by C. Schrage: *Set-Valued Convex Analysis*, 2009

||► **Thank you for your attention.**