Why Bankers Should Learn Convex Analysis

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Part 2: Pricing financial derivatives

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Summary

Convex analysis use to play a central role in economics and finance via (concave) utility functions.

A ‘new’ paradigm was emerged since the 1970’s after Black-Scholes introduced the replicating portfolio pricing method for option pricing, and Cox and Ross developed the risk neutral measure pricing formula.

This new paradigm marginalized many time tested empirical rules of the practitioners (based on concave utility optimization).

It brought about unprecedented prosperity in financial industry yet eventually led to the 2008 crisis.
We will show that the ‘new paradigm’ is a special case of the traditional utility maximization and its dual problem.

Overlooking sensitivity analysis in the ‘new paradigm’ is one of the main problem.

The recent financial crisis is a wake up call that it is time again for bankers to learn convex analysis.
The talk is divided into two parts. In the first part we discuss:

- A discrete model for financial markets.
- Arbitrage and martingale.
- Fundamental theorem of asset pricing.
- Utility functions and risk measures.
- Markowitz portfolio theory
The second part focuses on the financial derivatives.

- The new paradigm of financial derivative pricing.
- A Convex Analysis Perspective.
- Sensitivity and Financial Crisis.
- Alternative methods and an illustrative example using real historical market data.
The New Paradigm

Part 2: Pricing financial derivatives

Why Bankers Should Learn Convex Analysis
There are two basic types: \textit{calls} and \textit{puts}.

An call (put) option is a right to buy (sell) the stock at strike $K$ on Maturity.

Value at maturity when stock worth $S$:
- Call: $(S - K)^+ := \max(S - K, 0)$;
- Put: $(S - K)^- := \max(K - S, 0)$;

How much should it worth NOW?
Consider a stock whose current (spot) price is 1 and an call option with a strike 1 (at the money).

Assume the following:

- Stock Maturity
  - Stock: 1/2
  - Maturity: 2

- Option Maturity
  - Option: 1/2
  - Maturity: 1

How should we value current price $c$ of the option?
Consider a stock whose current (spot) price is 1 and an call option with a strike 1 (at the money).
Assume the following:

\[
\begin{array}{ccc}
\text{Stock} & \text{Maturity} \\
1 & 1/2 & 2 \\
\end{array}
\quad
\begin{array}{ccc}
\text{Option} & \text{Maturity} \\
c & 1/2 & 1 \\
\end{array}
\]

How should we value current price \(c\) of the option?
It is tempting to use the expected value of the option, which is 0.5 in this case, but....
One period model

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<thead>
<tr>
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<th>Option</th>
<th>Maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>0.5</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
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Black-Scholes’ replicating portfolio

They will spend \( p = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \) to assemble a portfolio (ignoring the fees):
\[
\frac{2}{3} \text{stock} - \frac{1}{3} \text{cash}
\]

which exactly replicates the outcome of the option. Thus, we should have \( c = p = \frac{1}{3} \). If \( c \neq p = \frac{1}{3} \) an arbitrage opportunity occurs.
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Note that the right mix for our portfolio is coming from solving

$$\begin{bmatrix} 2 \\ 0.5 \end{bmatrix} #stock + \begin{bmatrix} 1 \\ 1 \end{bmatrix} #cash = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
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In General

If $S$ is specified as

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</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$1 + a$</td>
<td>$c$</td>
<td>$a$</td>
</tr>
<tr>
<td>1</td>
<td>$1 - b$</td>
<td>$p_1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$p_2$</td>
<td></td>
<td>$p_2$</td>
<td></td>
</tr>
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</table>

Then the replicating portfolio is $a/(a+b)$ share of stock and $-a(1-b)/(a+b)$ share of cash and $c = ab/(a+b)$
Cox-Ross risk neutral measure

Shortly after, Cox and Ross proposed another ingenious idea which became prevalent in pricing financial derivatives.

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- Thus all the assets will have the same return.
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- If the option can be replicated then the above price equals the price in the real world.
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Calculating the risk neutral probability

The risk neutral probability (which usually differs from the actual probability) must satisfy the following two relations

- Stock return indifferent from that of (riskless) cash:
  \[ 1 = \pi_1(1 + a) + \pi_2(1 - b). \]

- Be a probability measure:
  \[ 1 = \pi_1 + \pi_2. \]

- Solving these two equations simultaneously we have
  \[ \pi_1 = \frac{b}{a + b}, \text{ and } \pi_2 = \frac{a}{a + b}. \]
Calculating the option price

In this risk neutral world the option price should also equal to the expected return.

\[ c = a \cdot \pi_1 + 0 \cdot \pi_2 = \pi_1 = \frac{ab}{a + b}. \]

This matches the price derived from using the portfolio replication method.
Remark

Both methods are based on the assumption that the option can be exactly replicated.

This almost never happens in practice.
If the price of option is \( c \), what should be the investment strategy for stock, cash and option given a utility function \( u \), (say the log utility \( u(t) = \ln t \) for \( t > 0 \) and \( -\infty \) otherwise)? Assuming unit initial endowment the problem is

\[
p = \max E \left[ u(\alpha S + \beta (S - 1)^+ + \gamma) \right],
\]

\[
\alpha + \beta c + \gamma = 1.
\]

For utility (such as \( \ln \)) that satisfies (u1)-(u4), it is not hard to show there is arbitrage iff \( p = +\infty \).
Define \( f(X) = E[-u(X)] \). We can rewrite problem as

\[
\min f(X), \quad s.t. \quad X = \alpha(S - 1) + \beta((S - 1)^+ - c) + 1;
\]

or

\[
(P) \quad \min f(X) + \nu_{1+K}(X)
\]

where \( K = \text{span}(S - 1, (S - 1)^+ - c) \).
It is easy to verify

\[(CQ) \quad 0 \in \text{int}(1 + K - R^2_+).\]

Moreover, \(\iota^*_{1+K}(z) = \langle 1, z \rangle + \iota_{K^\perp}.\) Thus, the minimum of \(P\) is equal to the maximum of the dual problem

\[(D) \quad \max \{\langle 1, z \rangle - f^*(z) : z \in K^\perp\}\]

\[= \max \left\{ z_1 + z_2 - \left( p_1(-u)^* \left( -\frac{z_1}{p_1} \right) + p_2(-u)^* \left( -\frac{z_2}{p_2} \right) \right) : z \in K^\perp \right\}\]
For \( u(t) = \ln t \) for \( t > 0 \) and \( -\infty \) otherwise, \((-u)^*(z) = 1 + \ln(z)\) for \( z > 0 \) and \( +\infty \) otherwise. This forces the solution \( z^* \) to \((D)\) in \( \text{int} \ R^2 \). Define \( \pi = -z^*/(z_1^* + z_2^*) \). We have

\[
\langle \pi, 1 \rangle = 1, \quad \pi \text{ is a measure,}
\]

\[
\langle \pi, S \rangle = 1, \quad \text{risk neutral,}
\]

and

\[
\langle \pi, (S - 1)^+ \rangle = c, \quad \text{pricing.}
\]
Financial derivative

Given a set of assets $\mathcal{A}$. A financial derivative is a random variable whose payoff is a function of that of the assets in $\mathcal{A}$.

- Options;
- Insurance;
- Credit default swaps (CDS) [Insurance on bonds];
- Collateral debt obligations (CDO) [Investing on trenches of pooled bonds].

Reminder: they can rarely be replicated.
Model financial derivatives

- Use $H \in RV(\Omega)$ to represent the payoff of the financial derivative.
- Let $H_0$ be the price of $H$.
- Assume that $H$ can only be traded at $t = 0$ and $t = T$.
- Then a self-financing trading strategy for $\mathcal{A} \cup \{H\}$ has the form $(\Theta, \beta)$ where $\Theta \in T(\mathcal{A})$ and $\beta \in R$ representing the share of $H$ in the trading strategy.
- Between 0 and $T$, $\beta$ is a constant.
Maximizing utility

Assuming an initial wealth $w_0 = 1$, we face the optimization problem

$$\max \quad (E(u(y)))$$

subject to

$$y \in G_T(T(A)) + \beta(H - H_0) + 1.$$ 

where $u$ is a utility satisfying assumptions (u1)-(u4).
Define \( f(y) = -E(u(y)) \) and \( g(y) = \nu G_T(T(A)) + \beta(H-H_0) + 1(y) \).

Problem (1) becomes

\[
- \min_y \{ f(y) + g(y) \} 
\]

(2)

The dual problem is,

\[
- \max \{-f^*(z) - g^*(-z)\} 
\]

(3)

\[
= \min \; \sigma_1 + \beta(H-H_0) + G_T(T(A))(-z) + E[(-u)^*(z)] 
\]

\[
= \min \; \{ \langle 1 + \beta(H-H_0), -z \rangle + \nu G_T(T(A)) \perp (z) \mid z > 0 \} 
\]

\[
= \min \; \{ -\langle 1, z \rangle \mid z > 0, \; z \in G_T(T(A)) \perp, \; \langle z, H-H_0 \rangle = 0 \}. 
\]
There is no arbitrage trading strategy for a financial market $\mathcal{A} \cup \{H\}$ if and only if

$$H_0 \in \{E^Q(H) \mid Q \in \mathcal{M}(\mathcal{A})\}.$$ 

Here $\mathcal{M}(\mathcal{A})$ is the set of risk neutral (martingale) measure on market $\mathcal{A}$ and $H$ is the payoff of the derivative.
Risk neutral pricing: Proof

There is no arbitrage for $\mathcal{A} \cup \{H\}$ iff the value of the optimization problem (1) is finite. Since $\text{CQ } 1 \in \text{dom } g \cap \text{int}(\text{dom } f)$ is true, the dual problem (3) also has a finite value. This happens iff, there exists $z > 0$

$$z \in G_T(T(\mathcal{A}))^\perp, \langle z, H - H_0 \rangle = 0. \quad (4)$$

Define $Q = z P / \int_\Omega z dP$. Then $Q \in \mathcal{M}(\mathcal{A})$ and $E^Q(H - H_0) = 0$. Thus, no arbitrage iff

$$H_0 \in \{E^Q(H) \mid Q \in \mathcal{M}(\mathcal{A}) \}. \quad (5)$$
Remarks

- In general martingale measures are not unique unless the market is ‘complete’.
- Thus we can only get a range of the price $H_0$.
- Additional assumptions are needed to determine a price.
- e.g. using entropy maximization has been proposed by Borwein et al. some time ago and gained attention recently.
- Selecting a particular risk neutral measure relates to change in utility.
- However, what is the change in the utility is not always clear.
Sensitivity

- Perceived model:
  - Stock: $\pi$, Maturity: $1 + a$
  - Option: $\pi$, Maturity: $a$

- Actual outcome:
  - Stock: $\pi$, Maturity: $1 + a + da$
  - Option: $\pi$, Maturity: $a + da$

Part 2: Pricing financial derivatives
Arbitrage position: sell one call option at \( \frac{ab}{a+b} + dc \) and buy \( \frac{a}{a+b} \) shares of stock at 1

Payoff:

<table>
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<th>Cost</th>
<th>Payoff</th>
<th>Percentage gain</th>
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</thead>
<tbody>
<tr>
<td>( \frac{a(1-b)}{a+b} - dc )</td>
<td>( \frac{a(1-b) + ada}{a+b} )</td>
<td>( \frac{ada + (a+b)dc}{a(1-b) - (a+b)dc} )</td>
</tr>
<tr>
<td>( 1 - \pi )</td>
<td>( \frac{a(1-b) - adb}{a+b} )</td>
<td>( \frac{-adb + (a+b)dc}{a(1-b) - (a+b)dc} )</td>
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Since typically \(|da|, |db| > dc\) the perceived arbitrage may turn out to be a pure losing position.
Why not the new paradigm?

- As a special case of the utility maximization that goes back at least to Bernoulli, the ‘new paradigm’ is not new.
- The assumptions are not particularly realistic or sound (unlimited leverage, no transaction cost, etc...).
- The log utility is known to be a bad choice for investment problems due to the instability of the solution.
- Sensitivity is largely neglected.
Financial Crisis

Sadly, the theoretical flaw alluded to above actually happens in the markets and in massive scale in 2008.

- First a small sector *subprime mortgage* gets in trouble causing a deviation from model.
- Expected arbitrage profit became loss.
- **Over leveraging** compounded the scale of the problem.
- The dominance of the ‘new paradigm’ means problem is universal so that *unloading the losing positions were impossible.*
Over leverage

- The estimated level of leverage for those falling giants are 30-40 times.
- Ten years ago such a leverage brought down LTCM.
- Twenty years ago it was the savings and loan crisis.
- The only justification for such level of leverage would be ‘riskless’.
- It seems that the illusion of arbitrage is at the root of all these crisis.
Motivation

- If the replicating portfolio pricing is instable, and
- yet largely followed in the market by majority of players.
- Then one should be able to take advantage of the situation.
Will convex analysis help? Dr. Anirban Dutta and I conducted ex-ante experiments using US historical option trading data. We

- choose a well known trend following method to practitioners;
- design a stable trading strategy base on consideration of worst case scenario;
- use a class of risk-reward functions that combines utility and risk measure to model tradeoff between gain and risk.
Worst Case

- Usually, $|da|, |db| \leq \delta$.
- The worst case scenario

\[
\begin{align*}
\text{Cost} & = \alpha + \beta(c + dc) \\
\text{Percentage gain} & = \frac{a(\alpha + \beta \frac{a}{a+b}) - |\alpha + \beta|\delta - \beta dc}{\alpha + \beta(c + dc)} \\
& \quad - \frac{-b(\alpha + \beta \frac{a}{a+b}) - |\alpha|\delta - \beta dc}{\alpha + \beta(c + dc)}
\end{align*}
\]

- Note that the percentage gains are homogeneous with respect to $(\alpha, \beta)$ and proportionally change the percentage gain (loss) yields equivalent portfolio.
Get the best out of the worst

- Trying to get the best expected return lead to the optimization problem

$$\max f(\alpha, \beta) := e(\alpha + \beta \frac{a}{a + b}) - (\pi |\alpha + \beta| + (1-\pi)|\alpha|)\delta - \beta dc,$$

Subject to $|\alpha| + |\beta| = 1$.

- Since $f$ is piecewise linear the candidates for solutions are the corner points of the constraint set $|\alpha| + |\beta| = 1$ and those satisfy $\alpha + \beta = 0$ and $\alpha = 0$. 

Part 2: Pricing financial derivatives

Why Bankers Should Learn Convex Analysis
An option replacement strategy

- A quick enumeration leads to three possibilities
  
  $f(1/2, -1/2) = \frac{eb}{a+b} + dc - (1 - \pi)\delta$: write a covered call option,
  
  $f(0, 1) = \frac{ea}{a+b} - dc - \pi\delta$: buy a call option, and
  
  $f(1, 0) = e - \delta$: hold the stock.

- Thus, a robust strategy is to do only one of these at a time. The question is which one?
Comparing

- We treat them all as investment systems represented by a random variable $x$ determined by the market historical return.
- and compare them with a risk reward function

$$K(x) = E[\ln(x)] - \nu_M(CV@R_\alpha(x)),$$

where $CV@R_\alpha$ is the conditional value at risk, a risk measure proposed by Rockafellar and Uryasev.
Let $x$ be the stock investment system. Then

- buying at money call option with a premium $p$ is characterized by

$$c(x, p) = \frac{x^+ - p}{p}$$

and

- writing an at money covered call option is

$$w(x, p) = \frac{p - x^-}{1 - p}.$$
Consider

\[ x = \{0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0, -1\}. \]

Graphing \( K(x) \), \( K(c(x, p)) \) and \( K(w(x, p)) \) together.

Graphing an explicit example
The Option Replacement Strategy

- $p < p^C$: Buy the call option
- $p^C \leq p \leq p^W$: Buy the stock
- $p > p^W$: Write the call option
Relationship with arbitrage pricing

- If the market is complete then \( p_C = p_W \) and the common value coincide with the arbitrage pricing by replicating portfolio.

- In this case, there is no difference in pricing.

- However, the trading strategy is different. The option replacement trading strategy is robust.
A Trend Following Investment System

- Standard and Poor’s 500 Index
- Invest only when market is going up
- Up trend: 40-day ema > 200-day ema
Return Distribution

- We use 60-months’ empirical distribution of discounted return
Finding a Stable Investment Size

- Investment size (leverage) is determined by optimizing the risk reward function.
- Parameters $M$ and $\alpha$ controls the tradeoff between return and risk.
Option to use

- We use at-money options \((a = 0)\)
- Use nearest in-the-money for buying call.
- Use nearest out-of-the-money for writing call.
Mild control on risk $M = 0.9$ and $\alpha = 0.1$
Testing Results

Tighter control on risk $M = 0.9$ and $\alpha = 0.05$
Past performance does not guarantee future profitability ......

- Theoretical robustness is base on the assumption that past and future behavior of the markets are similar.

- This may be true but....
Past performance does not guarantee future profitability ......

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Would they embrace the new paradigm? They certainly should not and as a result the financial crisis would be avoided. Therefore, it is high time to include convex analysis in the curriculum of financial engineering.
End of the Commercial!

Now the Movie
In the shoes of a ‘quant’

We have been discussing the issue from a convex analyst perspective.

Now let’s take the view of a quant. They know what Wall Street needs:

- Increasing trading volume (uniform pricing);
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We have been discussing the issue from a convex analyst perspective. Now let’s take the view of a quant. They know what Wall Street needs:

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- Play other people’s money and play big.

They have scored excellently so far with the help of the new paradigm.
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Let’s help the regulators.
Thank You!