Cogrowth

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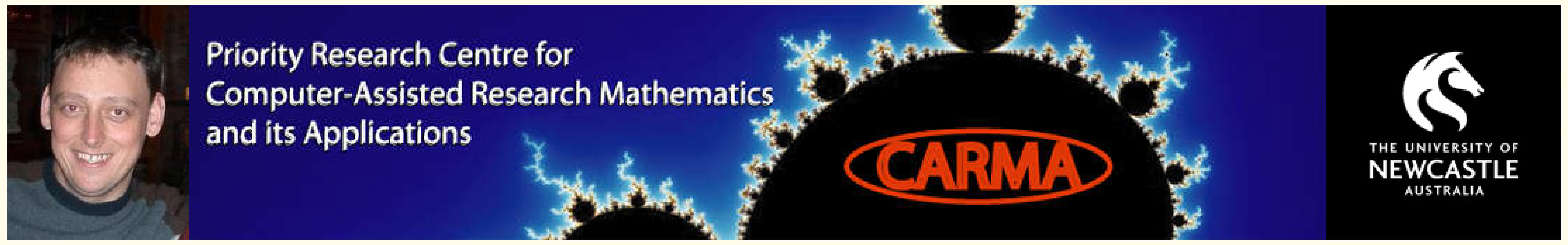
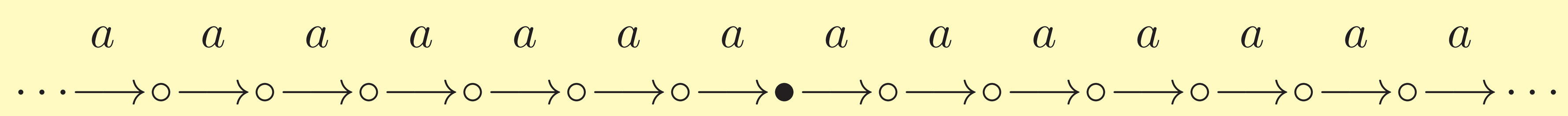


Walks in a graph

Consider a directed, edge-labeled graph, like:



A *walk* on the graph can be expressed as a string of as (going forwards along an a edge) and a^{-1}s (going backwards along an a edge). So the number of walks in this graph of length n, starting from the black node, is 2^n.

A *return* is a walk starting *and ending* at the black node. The number of returns in this graph of length n is 0 if n is odd, and $\binom{n}{n/2}$ if n is even (choose where the as go in a string of length n).

Asymptotically, $\binom{n}{n/2} \sim 2^n$ (think Catalan numbers), so the number of returns is asymptotically the same as the total number of walks.

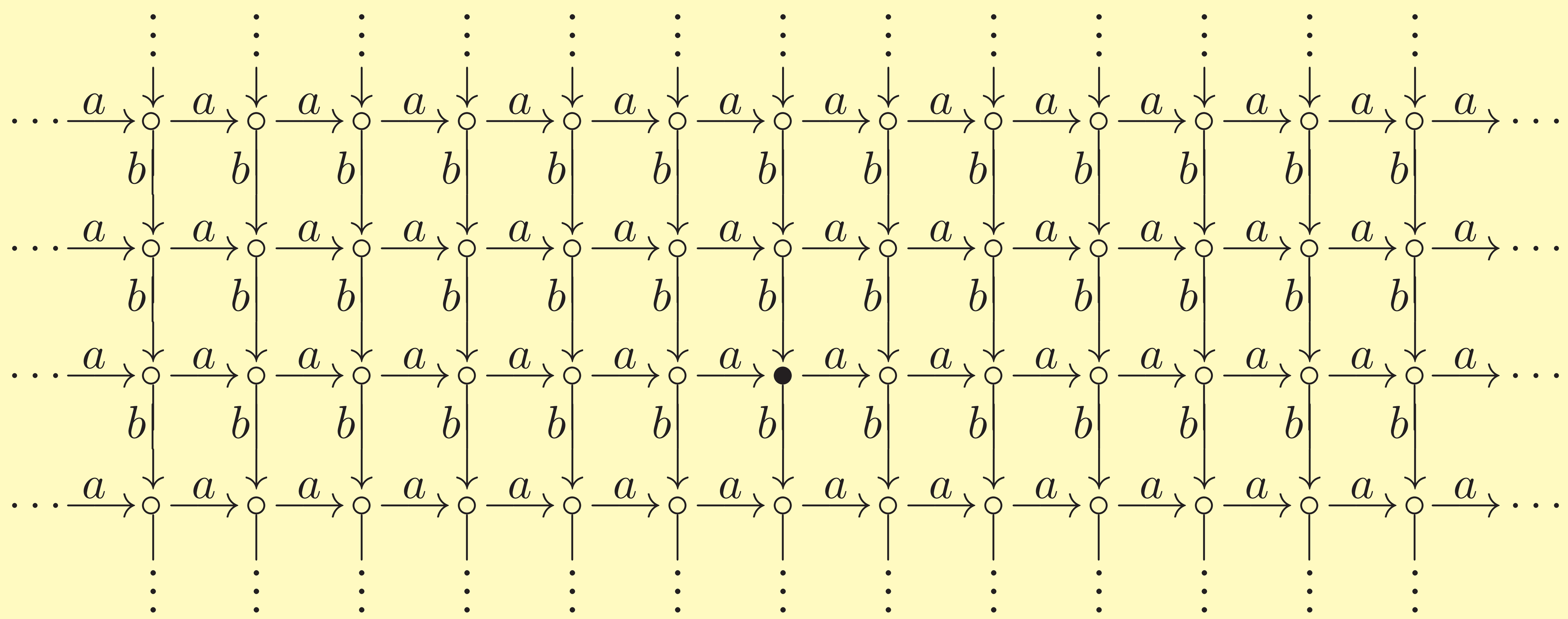
Cogrowth

If G is a *group* with a finite set of generators X, one can consider the *Cayley graph* of G, which is a directed, edge-labeled graph such that each node has an incoming and an outgoing edge labeled a for each $a \in X$, and with a distinguished node. In such graphs, the number of walks starting at this node is $(2|X|)^n$.

If r_n = the number of returns of length n, then $r_n r_k \leq r_{n+k}$ since $r_n r_k$ counts the returns of length $n+k$ that return at steps n and $n+k$. Then by Fekete's lemma [4], $\rho = \limsup_{n\to\infty} r_n^{1/n}$ exists. This constant is called the *cogrowth* of the Cayley graph of G.

Since the number of all walks in the Cayley graph is $(2|X|)^n$, an upper bound for ρ is $2|X|$. Grigorchuk [3] (and independently Cohen [1]) proved that $\rho = 2|X|$ if and only if the group is *amenable*, an important and much studied property in group theory.

Another (amenable) example



This is the Cayley graph of $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$ (think $a = (1,0)$ and $b = (0,1)$ with addition).

The number of walks (strings of $a^{\pm 1}, b^{\pm 1}$) from the black node is 4^n, and the number of returns is 0 if n is odd, else (sequence A002894 [5]) $\binom{n}{n/2}^2 \sim 4^n$, so $\rho = 4 = 2|X|$.

A nonamenable example

The *free group* on two letters a, b is the set of all strings of $a^{\pm 1}, b^{\pm 1}$ with no cancelling pairs $aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b$. Its Cayley graph is the 4-regular infinite tree.

An exact formula for its cogrowth series can be obtained, and $\rho \approx 3.464$. Note that this give a *lower bound* for the cogrowth of any group with two generators, since the Cayley graph of such a group has at least this many returns.

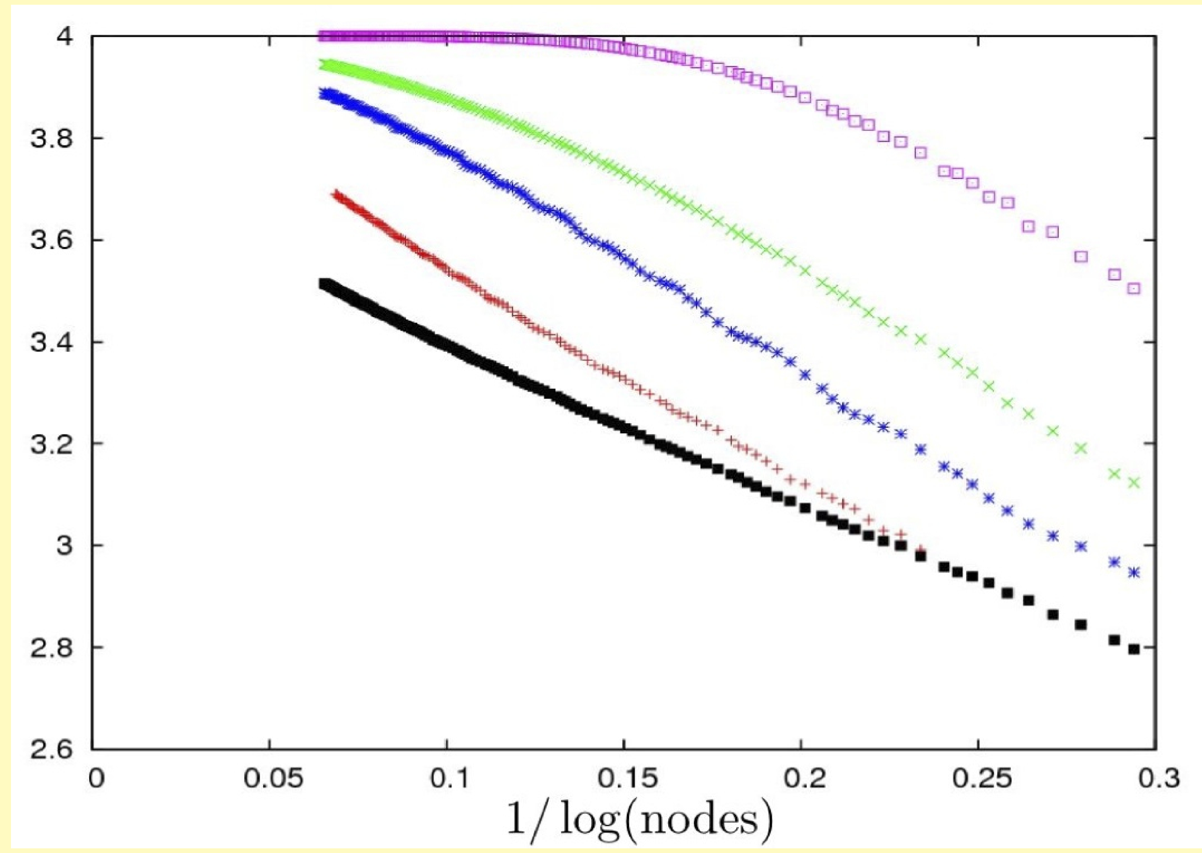
Computations

Since cogrowth is such a computational and combinatorial property, we decided to use it to tackle an open problem concerning the amenability of a particular example, Richard Thompson's group F.

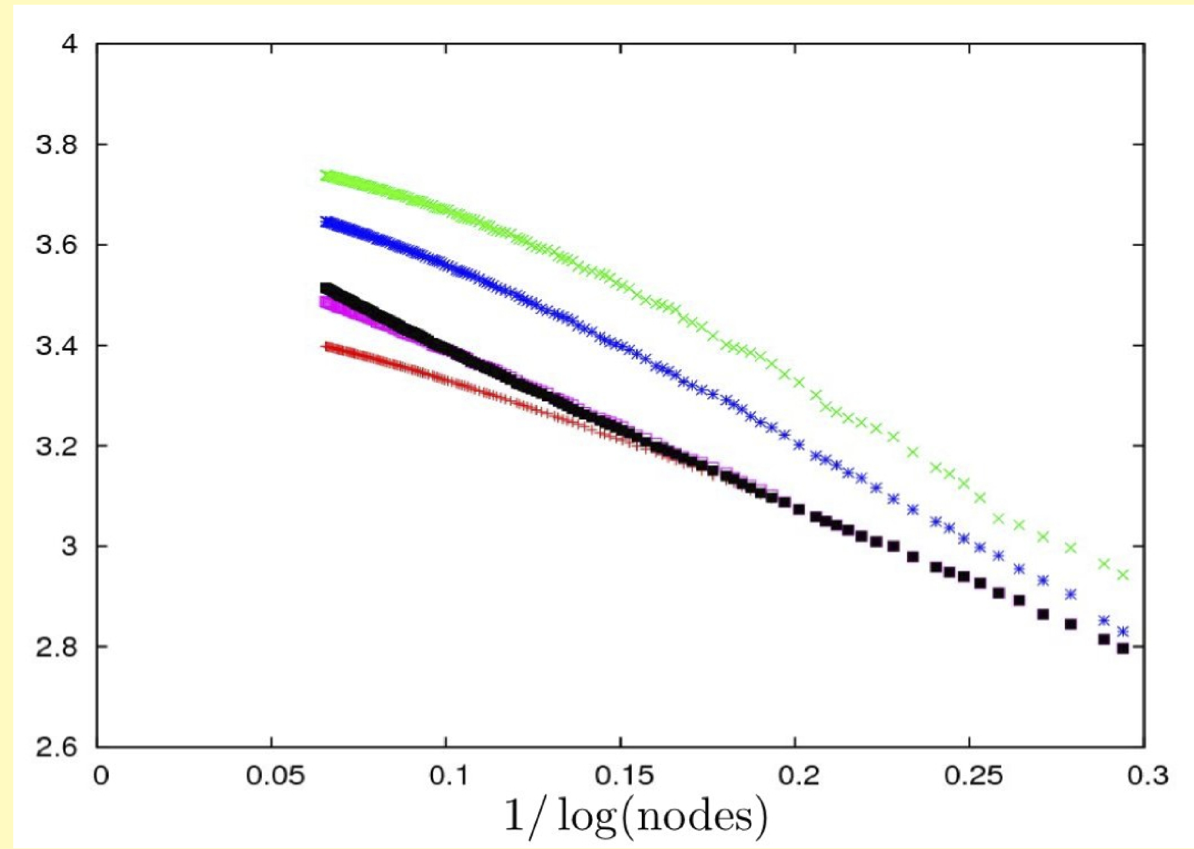
As an experiment, we computed bounds on the cogrowth of a number of different groups, some amenable and some nonamenable, and compared their behaviour against that of Thompson's group F, to see if it looks more like an amenable group or a nonamenable one.

If A is the *adjacency matrix* of the Cayley graph, then $(A^n)_{1,1}$ is the number of returns of length n. Of course A is infinite, but we can get a lower bound on r_n by taking a finite subgraph. If A_k is the adjacency matrix for a subgraph with k nodes, then (by Perron-Frobenious) its leading eigenvalue gives a lower bound for ρ.

We computed this eigenvalue for various groups with two generators. The horizonal axis is the size of the subgraph on a logarithmic scale, and the vertical axis is the eigenvalue. As k increases (so $\log(1/k) \to 0$) the lower bounds approach the real cogrowth rate ρ for each of the groups.



F (black) vs. amenable groups



F (black) vs. non-amenable groups

It is hard to conclude from this preliminary data which way F will go. We are currently working on better algorithms to produce more data.

References

[1] J. M. Cohen, Cogrowth and amenability of discrete groups *J. Funct. Anal.*, 48(3):301–309, 1982.
[2] M. Elder, A. Rechnitzer, T. Wong, On the cogrowth of Thompson's group F. Preprint on arXiv soon.
[3] R. Grigorchuk, Symmetrical random walks on discrete groups. *Adv. Probab. Related Topics* 6, pages 285–325. Dekker, 1980.
[4] J. H. van Lint, and R.M. Wilson, *A course on combinatorics*. Cambridge Univ Pr, 2001.
[5] The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2011.