Projection methods in geodesic metric spaces

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The feasibility problem associated with nonempty closed convex sets $A$ and $B$ is to find some $x \in A \cap B$. Projection algorithms in general aim to compute such a point.

These algorithms play key roles in optimization and have many applications outside mathematics - for example in medical imaging.

Until recently convergence results were only available in the setting of linear spaces (more particularly, Hilbert spaces) and where the two sets are closed and convex.

The extension into geodesic metric spaces allows their use in spaces where there is no natural linear structure, which is the case for instance in tree spaces, state spaces, phylogenomics and configuration spaces for robotic movements.
Diagram

Banach Spaces

CAT(0) spaces

Hilbert spaces

\( \ell_\infty \)

Hyperconvex metric spaces

R trees

Metric spaces
Definition

A subset $C$ of a geodesic metric space is **convex** if whenever $x, y$ are in $C$ every metric segment from $x$ to $y$ also lies in $C$. 
A geodesic metric space $X$ is a $\text{CAT}(\kappa)$-space if every geodesic triangle in $X$ satisfies the $\text{CAT}(\kappa)$-condition (inequality) relative to its comparison triangle in the (comparison) space $\mathbb{M}^2_\kappa$. 

comparison triangle in $\mathbb{E}^2$
\textbf{CAT}(\kappa) \text{ condition}

Here the \textit{comparison spaces} are:

\[ M^2_\kappa = \begin{cases} 
S^2_\kappa, & \text{if } \kappa > 0; \\
E^2, & \text{if } \kappa = 0; \\
H^2_\kappa, & \text{if } \kappa < 0. 
\end{cases} \]

Where:

\textit{Ian Searston} \hspace{1cm} \textit{Proj methods in CAT(0)}
CAT($\kappa$) condition

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E^2, & \text{if } \kappa = 0; \\
H^2_\kappa, & \text{if } \kappa < 0.
\end{cases}$$

Where:
$S^2_\kappa$ is the 2-sphere of radius $\frac{1}{\sqrt{\kappa}}$, 
$E^2$ is two dimensional Euclidean space, and 
$H^2_\kappa$ is the hyperbolic two manifold of constant negative curvature $\kappa$, 

\[ \begin{align*} 
\Omega_+&>1 \\
\Omega_-&<1 \\
\Omega_0&=1 
\end{align*} \]
We identify $\mathbb{H}^2_\kappa$ with the Poincaré upper half-plane \( \{ z \in \mathbb{C} : \Im z > 0 \} \) equipped with the metric \( \frac{1}{\sqrt{-\kappa}} d_P \) where

\[
d_P(z_1, z_2) = \int_{z_1}^{z_2} \frac{|dz|}{\Im z} = \cosh^{-1} \left( 1 + \frac{|z_1 - z_2|^2}{2\Im z_1 \Im z_2} \right).
\]

In which case, geodesics are semicircles with centres on the extended real axis.
Poincaré’s upper half-plane model for $\mathbb{H}^2_\kappa$

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In which case, geodesics are semicircles with centres on the extended real axis.

The geodesic through the points $z_1$ and $z_2$ in $\mathbb{H}^2_1$. 
CAT($\kappa$) spaces with $\kappa \leq 0$ display much of the geometry inherent in Euclidean space with geodesics playing the role of lines.

Spaces with *curvature bounded below*; that is, spaces $X$ for which $\inf\{\kappa : X \text{ is a CAT}(\kappa) \text{ space}\} > -\infty$, have non-bifurcating geodesics and are important to us because if geodesics are extendable their extensions are unique.

We will be especially interested in spaces which are CAT(0). A class which includes all CAT($\kappa$)-spaces with $\kappa < 0$. 
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We will be especially interested in spaces which are CAT(0). A class which includes all CAT($\kappa$)-spaces with $\kappa < 0$. 
$X$ satisfies the CN-inequality of Bruhat and Titts. That is for any three points $x, y_1, y_2 \in X$, and noting that $y_0$ is the metric midpoint of $y_1$ and $y_2$ if

$$d(y_1, y_2) = d(y_1, y_0) + d(y_0, y_2) \text{ and } d(y_1, y_0) = d(y_0, y_2)$$

then

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

*cf* the parallelogram law.
Proposition

Let $X$ be a $\text{CAT}(0)$ space and $C$ be a convex subset which is complete in the induced metric. Then,

1. for every $x \in X$ there exists a unique point $P_C(x) \in X$ such that $d(x, P_C(x)) = d(x, C) := \inf_{y \in C} d(x, y)$;

2. if $y$ belongs to the geodesic segment $[x, P_C(x)]$ we have $P_C(y) = P_C(x)$;

3. for any $x \in X \setminus C$ and $y \in C \setminus P_C(x)$ we have $\angle_{P_C(x)}(x, y) \geq \frac{\pi}{2}$

4. $P_C$ is a firmly nonexpansive (in the sense of Bruck $^\dagger$) retraction onto $C$.

$^\dagger$ $T : X \to X$ is firmly nonexpansive if $d(Tx, Ty) \leq d(tx \oplus (1 - t)Tx, ty \oplus (1 - t)Ty)$ all $t \in [0, 1]$. 
For $x \in X$ and $G$, any geodesic through $x$, we define the function $\phi_G : X \to \mathbb{R}$ by

$$\phi_G(x_n) := d(x, P_G(x_n)).$$

In a CAT(0)-space a sequence $(x_n)$ is weakly convergent to $x$ if and only if $\lim_{n \to \infty} \phi_G(x_n) = 0$, for every geodesic $G$ containing $x$.
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(Reshetnyak’s Gluing Theorem.) Let \( \{(X_i, d_i)\}, i = 1, 2 \) be two complete spaces of curvature \( \leq k \). Suppose that there are convex sets \( C_i \in X_i \) and an isometry \( f : C_1 \to C_2 \). Attach these spaces together along the isometry \( f \).

Then the resulting space \( (X, d) \) is a space of curvature \( \leq k \).

We illustrate with an example.
The convex feasibility problem associated with the nonempty closed convex sets $A, B$ is to

“find some $x \in A \cap B$”.

Projection algorithms in general aim to compute such a point. We consider two such algorithms in the context of CAT(0) spaces.

This allows us to treat feasibility problems where the sets are metrically, but not necessarily algebraically, convex. For example star shaped sets in $E^2$. 
The method of alternating projection into convex sets (sometimes known as "project, project") emerged from initial work by John von Neumann (1903 – 1957) who, in the 1930s, proved that when $A$ and $B$ were closed affine manifolds of a Hilbert space the iterative scheme $x_{n+1} = P_B P_A x_n$ converged in norm for any initial starting point $x_0$ to $P_{A \cap B} x_0$. 

![Diagram showing norm convergent to nearest point in $A \cap B$](image-url)
In 1965, weak convergence was established by L. M. Bregman when $A, B \in H$ are closed convex sets in a Hilbert space with $A \cap B \neq \emptyset$. Examples show that norm convergence need not occur.

The Hilbert space proof can be adapted to obtain an analogous result in CAT(0) spaces [Bacak, S & Sims].

**Theorem**

Let $X$ be a complete CAT(0) space and $A, B \subset X$ convex, closed subsets such that $A \cap B \neq \emptyset$. Let $x_0 \in X$ be a starting point and $(x_n)$ be the sequence generated by alternating projections. Then $(x_n)$ weakly converges to a point $x \in A \cap B$.

*Strong convergence pertains when $A$ and $B$ satisfy certain “regularity” conditions and various estimates on the rate of convergence are possible.*
Douglas-Rachford method

Starting with any initial point $x_0$, the Douglas-Rachford algorithm is the iterative scheme

$$x_{n+1} := T(x_n) \text{ where, } T = \frac{1}{2}(R_AR_B + I).$$
Provided $A$ and $B$ are convex and have a non-empty intersection, the Douglas-Rachford algorithm was shown to converge weakly to a point $x$ with $P_B x \in A \cap B$, by P-L. Lions and B. Mercier in 1979.

Pierre-Louis Lions  
Bertrand Mercier
Impediments to extending Douglas-Rachford into CAT(0) spaces:
How to define reflection?
How to show convergence?

To discuss reflections in CAT(0) spaces we require geodesics to be extendable. We also require that the extension is unique which happens if and only if the curvature is bounded below.
Impediments to extending Douglas-Rachford into CAT(0) spaces:
How to define reflection?
How to show convergence?
To discuss reflections in $CAT(0)$ spaces we require geodesics to be extendable. We also require that the extension is unique which happens if and only if the curvature is bounded below.
With the above conditions we can define the reflection of a point $x$ in a closed convex subset $C$ of $X$, a $CAT(0)$ space, to be a point $R_{C}(x)$ on a geodesic which is an extension of the segment $[x, P_{C}(x)]$ such that

$$d(R_{C}(x), P_{C}(x)) = d(x, P_{C}(x)),$$

where $P_{C}x$ is the projection of $x$ onto the set $C$. 

![Diagram showing the reflection process](image)
It is well known that reflections in Hilbert space are non-expansive; this follows since the closest point projection is firmly nonexpansive, something which is also true in an appropriate sense in CAT(0) spaces.
Using the appropriate “Law of cosines” Fernández-Leon – Nicolae [2012] proved the following.

**Proposition**

For $k \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose $C$ is a nonempty closed and convex subset of $M_k^n$ and $x, y \in M_k^n$ such that $\text{dist}(x, C), \text{dist}(y, C) < D_k/2$. Then,

$$d(R_Cx, R_Cy) \leq d(x, y).$$

Using this they go on to establish weak convergence of Douglas-Rachford in the classical spaces $M_k^n$ of constant curvature.

However, in general reflections in CAT(0) spaces need not be nonexpansive.
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However, in general reflections in CAT(0) spaces need not be nonexpansive.
To conclude we construct and investigate a special instance of a CAT(0) space of non-constant curvature. We begin with the CAT(0) space $\Phi$ consisting of the geodesic (convex subset) $|z| = 1$ in the Poincaré upper half-plane which we may identify with

$$\Phi = \left( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), d_P \right)$$

where $d_P$ is the restriction of the “Poincaré metric” given by

$$d_P(\phi_1, \phi_2) = \int_{\phi_1}^{\phi_2} \frac{d\phi}{\cos(\phi)} = \left[ \ln(\sec(\phi) + \tan(\phi)) \right]_{\phi_1}^{\phi_2}.$$ 

Since the function $\ln(\sec(\phi) + \tan(\phi))$ occurs frequently in what follows we will denote it by $H(\phi)$.
$X := \Phi \otimes_2 \mathbb{E}^1$ (\(X_+ := \Phi \otimes_2 |\mathbb{E}^1|\)); the \(\ell_2^2\)-direct product of \(\Phi\) with \(\mathbb{E}^1 - 1\)-dimensional, Euclidean space (\(|\mathbb{E}^1| - \) the positive cone in 1-dimensional, Euclidean space), and metric given by,

$$d_X((\phi_1, h_1), (\phi_2, h_2)) = \sqrt{(d_P(\phi_1, \phi_2))^2 + (h_1 - h_2)^2}$$

$$= \sqrt{(H(\phi_2) - H(\phi_1)^2 + (h_1 - h_2)^2},$$

for \(h_1, h_2 \in \mathbb{R} (\geq 0)\) and \(-\pi/2 < \phi_1, \phi_2 < \pi/2\).
NOTE: \( X \) and \( X_+ \) are constant curvature (flat; curvature 0) CAT(0) spaces.
The unique geodesic $\Gamma$ in $X$ (or $X_+$) passing through two distinct points $P_1 : (\phi_1, h_1)$ and $P_2 : (\phi_2, h_2)$ is:

- the ‘vertical’ line (half line) $\{(\phi, h) : h \in \mathbb{R}(h > 0)\}$ if $\phi_1 = \phi_2$,
- otherwise, using the Euler-Lagrange equation to minimize the length of a curve from $P_1$ to $P_2$, we find $\Gamma$ has equation,

$$h(\phi) = AH(\phi) + B,$$

where the constants $A$ and $B$ are uniquely determined from the condition that $P_1, P_2 \in \Gamma$, in particular $A = \frac{h_1 - h_2}{H(\phi_1) - H(\phi_2)}$. 
A geodesic in $X := \Phi \otimes_2 \mathbb{E}^1$
Poincaré revisited

In the upper half plane model of the hyperbolic space $H^2_{-1}$ let $Y = \{z : \Im z > 0, |z| \leq 1\}$ equipped with the metric $d_P$ inherited from $H^2_{-1}$. $Y$ is a closed, convex subset of $H^2_{-1}$ and hence a $CAT(0)$ space of constant curvature $-1$.

Let $C$ be the geodesic in $Y$ given by $C = \{e^{i\theta} : 0 < \theta < \pi\}$. Then, $C$ is also a closed, convex subset of $Y$ and under the mapping $\phi \mapsto e^{i(\frac{\pi}{2} - \phi)}$, $\Phi$ is isometric to $C$. 
The space $Z$

$Z$ is obtained by gluing $X_+$ to $Y$ under the identification of $\Phi$ with $C$ which by Reshetnyak’s gluing theorem is a CAT(0) space of non-constant curvature, bounded below by $-1$.

Geodesics in $Z$ are uniquely extendable, and so reflection in closed convex sets of $Z$ is well defined.
A model for $Z$ as a submanifold of $\mathbb{E}^3$
An upper-half plane model for $Z$

We model $Y$ in the upper half-plane as above and identify points in $X_+$ with points in $W := \{ \rho e^{i\theta} : \rho \geq 1, 0 < \theta < \pi \}$ under the mapping

$$(\phi, h) \mapsto (1 + h)e^{i(\frac{\pi}{2} - \phi)}.$$ 

This naturally identifies $\Theta$ with $C$ and is an isometry when $W$ is equipped with the metric,

$$d_W(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2} = \sqrt{(R(\theta_2) - R(\theta_1))^2 + (\rho_2 - \rho_1)^2},$$

where $R(\theta) := H(\frac{\pi}{2} - \theta) = \ln(\text{cosec}(\theta) + \cot(\theta))$. 

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**Diagram Description:**

- The upper half-plane $Y$ is shown with points $\theta$ and $1$.
- The mapping $H(\frac{\pi}{2} - \theta)$ is depicted with a logarithmic function, showing the transformation from $X$ to $W$.
- The metric $d_W$ is illustrated, emphasizing the distance between points $\rho_1 e^{i\theta_1}$ and $\rho_2 e^{i\theta_2}$.
Some geodesics in \( \mathbb{Z} \)

Some geodesics in the upper half-plane model of \( \mathbb{Z} \)
View of reflections in $C$

reflection in $C$
In general $R_C|_Y$ is nonexpansive, but $R_C|_W = (R_C|_Y)^{-1}$ need not be.

For instance, as illustrated in the previous slide, the points $P_1 = i/2$ and $P_2 = 0.5439 + 0.4925i$ in $Y$ have

$Q_1 := R_C(P_1) = 1.6931i$ and $Q_2 := R_C(P_2) = 1.453e^{\pi/4}$

and

$$d_W(Q_1, Q_2) = 0.9135 < d_Y(P_1, P_2) = 1.0476$$

and so,

$$d_Z(R_C(Q_1), R_C(Q_2)) = d_Y(P_1, P_2) > d_Z(Q_1, Q_2)$$
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and so,

$$d_Z(R_C(Q_1), R_C(Q_2)) = d_Y(P_1, P_2) > d_Z(Q_1, Q_2)$$
We take as our two convex sets in $\mathbb{Z}$ the closed half-rays $A = \{ e^{i\theta} : 3\pi/4 \leq \theta \leq \pi \}$ and $B = \{ e^{i\theta} : 0 \leq \theta \leq 3\pi/4 \}$ of $C$, so $A \cap B = \{ e^{3\pi i/4} \} = \{ -0.7071, 0.7071 \}$. 
The following table shows three iteration of Douglas-Rachford, starting from $x_1 = (0.5439, 0.4925)$; points $z$ in $Y$ are specified by $(\Re z, \Im z)$ and those in $X_+$ by $(\theta, h)$.

<table>
<thead>
<tr>
<th></th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>$0.5439, 0.4925$</td>
<td>$-0.607669, 0.625647$</td>
<td>$-0.624135, 0.613204$</td>
</tr>
<tr>
<td>$P_Bx_n$</td>
<td>$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$</td>
<td>$-0.690260, 0.723561$</td>
<td>$-0.707009, 0.707205$</td>
</tr>
<tr>
<td>$R_Bx_n$</td>
<td>$\left(\frac{\pi}{4}, 0.4530\right)$</td>
<td>$-0.761849, 0.190098$</td>
<td>$-0.785260, 0.190012$</td>
</tr>
<tr>
<td>$P_AR_Bx_n$</td>
<td>$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$</td>
<td>$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$</td>
<td>$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$</td>
</tr>
<tr>
<td>$R_AR_Bx_n$</td>
<td>$\left(-0.895023, 0.122895\right)$</td>
<td>$\left(-0.639941, 0.600583\right)$</td>
<td>$\left(-0.624326, 0.613055\right)$</td>
</tr>
<tr>
<td>$x_{n+1}$</td>
<td>$-0.607669, 0.625647$</td>
<td>$-0.624135, 0.613204$</td>
<td>$-0.624231, 0.613130$</td>
</tr>
</tbody>
</table>

The iterates appear to be rapidly stabilizing with $P_Bx_n$ converging to the feasible point.
The iterates from the alternative starting point \( y_1 = (0, 0.5) \) behave similarly.

<table>
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<th>( n = 2 )</th>
<th>( n = 3 )</th>
</tr>
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<tbody>
<tr>
<td>( y_n )</td>
<td>((0, 0.5))</td>
<td>((-0.509291, 0.485280))</td>
<td>((-0.533052, 0.474962))</td>
</tr>
<tr>
<td>( P_{By_n} )</td>
<td>((0, 1))</td>
<td>((-0.681383, 0.731927))</td>
<td>((-0.706154, 0.708058))</td>
</tr>
<tr>
<td>( R_{By_n} )</td>
<td>((0, 0.6931))</td>
<td>((-0.749650, 0.492872))</td>
<td>((-0.784052, 0.495575))</td>
</tr>
<tr>
<td>( P_{AR_{By_n}} )</td>
<td>((-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}))</td>
<td>((-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}))</td>
<td>((-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}))</td>
</tr>
<tr>
<td>( R_{AR_{By_n}} )</td>
<td>((-0.744747, 0.231161))</td>
<td>((-0.555758, 0.463751))</td>
<td>((-0.534775, 0.474034))</td>
</tr>
<tr>
<td>( y_{n+1} )</td>
<td>((-0.509291, 0.485280))</td>
<td>((-0.533052, 0.474962))</td>
<td>((-0.533914, 0.474499))</td>
</tr>
</tbody>
</table>

These two tables also show that in these instances the iterated map \( T := \frac{1}{2}(I + R_{AR_B}) \) is nonexpansive, even though some intermediary steps are not.
Specifically;

\[ d_Y(Tx_1, Ty_1) = 0.3098 \leq d_Y(x_1, y_1) = 1.0476 \]
\[ d_Y(Tx_2, Ty_2) = 0.3056 \leq d_Y(x_2, y_2) = 0.3098 \]
\[ d_Y(Tx_3, Ty_3) = 0.3056 \leq d_Y(x_3, y_3) = 0.3056 \]

While \( d(R_AR_Bx_1, R_AR_By_1) = 1.0500 > d(x_1, y_1) = 1.0476 \)!

Thus, while reflections in CAT(0) spaces of non-constant curvature need not be nonexpansive, it appears that the averaging process in Douglas-Rachford iteration may compensate for this. This seems deserving of further investigation.
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