Turnpike theorems for convex problems with undiscounted integral functionals

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• Turnpike theory

• Continuous time systems
  – Undiscounted integral functionals
  – Convex problems

• Turnpike Theorems
Turnpike Theory

Optimal control problem:

- System: \( x_{t+1} \in a(x_t), \ t = 0, 1, 2, \ldots \)

- Functional

  \[ \text{Maximize: } \sum_{t=0}^{T} u \]

  where \( u = u(x_t) \) or \( u = u(x_t, x_{t+1}) \).

**Turnpike property** describes the “structure/behaviour” of optimal solutions when \( T \to \infty \)

- \( \exists \) “turnpike set/set” that attracts all opt. solutions
• J.V. Neumann, 1932-1945 - first result obtained
  – 1932 - presented at a math.seminar at Princeton (D.Gale)
  – 1937 - published in Vienna
  – 1945 - translated into English

• P.A. Samuelson, 1948-1949 - Interpretation of Neumann’s result

• 1958 - the term Turnpike was introduced in
• A.M. Rubinov, 1973 - Classification of the turnpike property (linear systems - Neumann-Gale model)
  – V.L. Makarov and A.M. Rubinov, Mathematical theory of economic dynamics and equilibria, 1973 (Russian)
  – translated into English, 1977
• L. McKenzie, 1976 - Nonlinear systems (bounded trajectories)

Discrete Systems: the main result

**Turnpike property is true for convex problems**

*graph a is convex, u is strongly concave*
Continuous time systems

System: \( \dot{x} \in a(x) \)

Functional: Utility fun. - \( u(t) = u(x(t)) \) or \( u(x(t), \dot{x}(t)) \)

1. Discounted integral: \( \int_0^\infty u(t) e^{-rt} dt \)
2. Undiscounted integral: \( \int_0^T u(t) dt \)
3. Terminal: \( \lim \inf_{t \to \infty} u(t) \)

Main focus: Convex Problems

- \( \text{graph } a = \{(x, y) : x \in \Omega, y \in a(x)\} \Rightarrow \text{is convex;} \)
- \( u \Rightarrow \text{is strongly concave.} \)
Some existing approaches

- Jose A. Scheinkman (≥ 1976) in collaboration with W.A. Brock, A Araujo etc (Maximum Principle)
- D.E.Gusev and V.A.Yakubovich (≥ 1973) (Maximum Principle)
- A.I.Panasyuk and V.I.Panasyuk (applications in engineering)
- M.Marena and L.Montrucchio
Recent developments

- Long run average problem (V. Gaitsgory, 2006)
  $$\lim_{T \to \infty} \frac{1}{T} \int_0^T u(x(t))dt$$

- Markov Games (V. Kolokoltsov at all, 2013)

- Model predictive control (T. Damm, L. Grüne et al, 2012-2014) (discrete systems)

- Time-delay systems (A. Ivanov and M. Mammadov, \( \geq 2010 \))

- Weak stability:
  - Statistical convergence (S. Pehlivan and M. Mammadov, 2000)
  - \( A \)-Statistical convergence (P. Das, S. Dutta et all, 2014)
  - Ideal convergence (M. Mammadov and P. Szuca, 2014)
My target: to develop a complete theory for undiscounted and terminal functionals by considering
- non-convex problems
- convex problems

Today’s talk: convex problems with undiscounted functionals

- Optimality: Overtaking optimal solutions on $[0, \infty));
- The convex case still uses some restrictive assumptions.
Turnpike Theorems

Problem (P):

System: \( \dot{x} \in a(x), \ x(0) = x^0, \)

Maximize: \( J_T(x(\cdot)) = \int_0^T u(x(t)) \, dt \)

- \( a : \Omega \searrow R^n \) has compact images, is continuous in the Hausdorff metric
- \( u : \Omega \rightarrow R^1 \) is continuous
- \( X_T \neq \emptyset \) denotes the set of trajectories on the interval \([0, T]\)
- \( \Omega \) is bounded and \( x(t) \in \text{int} \Omega, \ \forall t \in [0, T], \ x(\cdot) \in X_T, \ T > 0 \)
- \( M \triangleq \{ x \in \Omega, \ 0 \in a(x) \} \) - is the set of stationary points
- \( x^* \in M \) is optimal stationary point if \( u(x^*) = \max_{x \in M} u(x) \)
- Given \( T > 0 \), trajectory \( x(\cdot) \) is called
  - optimal if \( J_T(x(\cdot)) = J_T^* \triangleq \sup J_T(x(\cdot)) \)
  - \( \xi \)-optimal if \( J_T(x(\cdot)) \geq J_T^* - \xi; \ \text{where} \ \xi \geq 0. \)
Main Assumptions

A1 (Exist. “good” sol-s): $\exists \ b < +\infty, \text{ for every } T > 0 \ \exists \ x(\cdot) \in X_T:$

$$J_T(x(\cdot)) \geq u^* T - b.$$  

A2 (Convex Problem):

- $\text{graph} \ \alpha$ is convex, compact
- $u$ is concave (not necessarily strictly)
- $\forall \ x_1, x_2 \in \Omega, \ \alpha \in (0, 1), \text{ one of the following holds:}$

$$u(\alpha x_1 + (1 - \alpha) x_2) > \alpha u(x_1) + (1 - \alpha) u(x_2);$$

$$\text{int} \ a(\alpha x_1 + (1 - \alpha) x_2) \supset \alpha a(x_1) + (1 - \alpha) a(x_2).$$

A3: There exists $x' \in \Omega$ such that $u(x') > u^*.$

A4: There exists $\tilde{x} \in M$ such that $0 \in \text{int} \ a(\tilde{x}).$
**Theorem 3.1:** Assume that Assumptions **A1-A4** hold. Then there exists a unique optimal stationary point \( x^* \) and

1. **Upper bound for** \( J_T(x(\cdot)) \): there exists \( C < +\infty \) such that
   \[
   \int_0^T u(x(t)) \, dt \leq u^* T + C
   \]
   for all \( T > 0 \) and for all trajectories \( x(\cdot) \in X_T \);

2. **Turnpike property:** (given any \( \xi \geq 0 \)): for every \( \varepsilon > 0 \), there exists \( K_\varepsilon < +\infty \) s.t.
   \[
   \text{meas}\{t \in [0,T] : ||x(t) - x^*|| \geq \varepsilon\} \leq K_\varepsilon
   \]
   for all \( T > 0 \) and for all \( \xi \)-optimal trajectories \( x(\cdot) \in X_T \);

3. if \( x(\cdot) \) is an optimal trajectory and \( x(t_1) = x(t_2) = x^* \), then
   \[
   x(t) = x^*, \forall t \in [t_1, t_2].
   \]
Two special cases.

- **Utility function** $u$ **is strictly concave:**
  
  **A3** can be eliminated: $\exists x' \in \Omega$ such that $u(x') > u^*$. 

  **Theorem 3.2:** Assume that function $u$ is strictly concave and Assumptions **A1, A2, A4** hold. Then there exists a unique optimal stationary point $x^*$ and all the assertions (1)-(3) of Theorem 3.1 are valid.

- **Mapping** $a$ **is strictly convex:**
  
  **A4** can be eliminated: $\exists \tilde{x} \in M$ such that $0 \in \text{int} \ a(\tilde{x})$

  **Theorem 3.3:** Assume that mapping $a$ is strictly convex, Assumptions **A1, A2, A3** hold. Then there exists a unique optimal stationary point $x^*$ and the assertions (2) and (3) of Theorem 3.1 are valid.
THANK YOU