Existence and Approximation of Fixed Points of Bregman Nonexpansive Operators in Reflexive Banach Spaces

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General Assumptions

X a reflexive Banach space

$f : X \to \mathbb{R}$ a proper, convex and l.s.c function which is Gâteaux differentiable

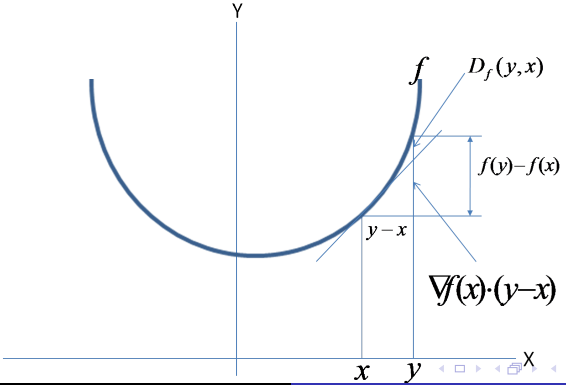
K a nonempty, closed and convex subset of X

Bregman Distance

Definition [Bregman (1967), Censor and Lent (1981)]

The **Bregman distance** $D_f : \operatorname{dom} f \times \operatorname{int} \operatorname{dom} f \to [0, +\infty)$ is defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle.$$



Classes of Operators in Hilbert Spaces

Let K be a nonempty, closed and convex subset of a Hilbert space H and let $T : K \to K$ be an operator

T is nonexpansive if for all $x, y \in K$

$$\|Tx - Ty\| \leq \|x - y\|$$

T is quasi-nonexpansive if for all $p \in F(T)$ and $x \in K$

$$\|p - Tx\| \leq \|p - x\|$$

T is firmly nonexpansive if for all $x, y \in K$

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle$$

Classes of Operators in Banach Spaces

Let $f : X \to \mathbb{R}$ be a function. Let K be a nonempty, closed and convex subset of X and let $T : K \to K$ be an operator

T is Bregman nonexpansive (BNE) if for all $x, y \in K$

$$D_f(Tx, Ty) \leq D_f(x, y)$$

T is quasi-Bregman nonexpansive (QBNE) if for all $p \in F(T)$ and $x \in K$

$$D_f(p, Tx) \leq D_f(p, x)$$

Definition [Bauschke, Borwein and Combettes (2003)]

T is Bregman firmly nonexpansive (BFNE) if for all $x, y \in K$

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle$$

or equivalently

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq$$

$$D_f(Tx, y) + D_f(Ty, x)$$

T is quasi-Bregman firmly nonexpansive (QBFNE) if for all

$p \in F(T)$ and $x \in K$

$$D_f(p, Tx) + D_f(Tx, x) \leq D_f(p, x)$$

Definition [Reich (1996)]

T is Bregman strongly nonexpansive (BSNE) with respect to a nonempty set S if

$$D_f(p, Tx) \leq D_f(p, x)$$

for all $p \in S$ and $x \in K$, and if whenever $\{x_n\}_{n \in \mathbb{N}} \subset K$ is bounded, $p \in S$, and

$$\lim_{n \to +\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n \to +\infty} D_f(Tx_n, x_n) = 0$$

Definition [Reich (1996)]

A point $u \in K$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in K such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. We denote the set of asymptotic fixed points of T by $\widehat{\mathrm{F}}(T)$.

Let $\{T_i : 1 \leq i \leq N\}$ be N BSNE operators and denote the composition $T_N T_{N-1} \cdots T_1$ by T and

$$F = \cap \{F(T_i) : 1 \leq i \leq N\} \text{ and } \hat{F} = \cap \left\{\hat{F}(T_i) : 1 \leq i \leq N\right\}$$

Properties of BSNE Operators

Lemma [Reich (1996)]

Let $f : X \to \mathbb{R}$ be bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of K. If the sets $\hat{F}(T)$ and $\hat{F}$ are nonempty, then T is BSNE with respect to $\hat{F}(T)$ and $\hat{F}(T) \subset \hat{F}$.

If $\hat{F}(T_i) = F(T_i)$ for each $1 \leq i \leq N$, F and $F(T)$ are nonempty, then T is also BSNE with $F(T) = \hat{F}(T)$. Indeed,

$$F(T) \subset \hat{F}(T) \subset \hat{F} = F \subset F(T)$$

Legendre Functions

Definition [Bauschke, Borwein and Combettes (2001)]

The convex function f is called a **Legendre function** if it satisfies the following two conditions:

(L1) The interior of the domain of f, $\operatorname{int}\operatorname{dom} f$, is nonempty, f is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f$;

(L2) The interior of the domain of f^*, $\operatorname{int}\operatorname{dom} f^*$, is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f^*$ and $\operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$.

Legendre Functions

Proposition [Bauschke, Borwein and Combettes (2001)]

f is Legendre if and only if f^* is Legendre.

Proposition [Bauschke, Borwein and Combettes (2001)]

If f is a Legendre function then:

- f is strictly convex on the interior of its domain.
- $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*$.

Note: $\nabla f^* = (\nabla f)^{-1}$.

Properties of the Gradient

Proposition [Reich and Sabach (2009)]

If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.

Proposition [Reich and Sabach (2010)]

If $f : X \to (-\infty, +\infty]$ is a positively homogeneous function of degree $\alpha \in \mathbb{R}$, then ∇f is a positively homogeneous function of degree $\alpha - 1$.

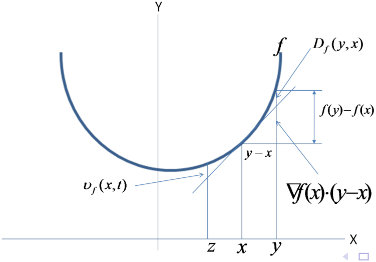
Totally Convex Functions

Definition [Butnariu and Iusem (2000)]

The function f is called **totally convex at a point** $x \in \operatorname{int}\operatorname{dom} f$ if its **modulus of total convexity at** x, that is, the function $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\upsilon_f(x, t) := \inf \{ D_f(y, x) \ : \ y \in \operatorname{dom} f, \ \|y - x\| = t \}$$

is positive whenever $t > 0$.



Totally Convex Functions

Proposition [Resmerita (2004)]

Let $f : X \to (-\infty, +\infty]$ be a function and take $x \in \operatorname{int}\operatorname{dom} f$.

Then f is totally convex at x if and only if $\lim_{n\to\infty} D_f(y_n, x) = 0$

implies that $\lim_{n\to\infty} \|y_n - x\| = 0$ for any sequence

$\{y_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} f$.

Characterization of Bregman Projections

Proposition [Butnariu and Resmerita (2006)]

Suppose that f is totally convex on $\mathrm{int}\,\mathrm{dom}\, f$. Let $x \in \mathrm{int}\,\mathrm{dom} f$ and let $K \subset \mathrm{int}\,\mathrm{dom} f$ be a nonempty, closed and convex set. If $\hat{x} \in K$, then the following statements are equivalent:

- The vector $\hat{x}$ is the Bregman projection of x onto K with respect to f;
- The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \qquad \forall y \in K;$$

- The vector $\hat{x}$ is the unique solution of the inequality

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \qquad \forall y \in K.$$

Lemma 1.3.1

Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let K be a nonempty, closed and convex subset of $\operatorname{int} \operatorname{dom} f$ and let $T : K \to K$ be a QBNE operator with respect to f. Then $\operatorname{F}(T)$ is closed and convex.

Lemma 1.3.2

Let $f : X \to (-\infty, +\infty]$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of X. Let K be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$ and let $T : K \to K$ be a BFNE operator with respect to f. Then

$$\mathrm{F}(T) = \widehat{\mathrm{F}}(T)$$

Theorem 1.4.1

Let $f : X \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of X. Let K be a nonempty, closed and convex subset of $\mathrm{int}\,\mathrm{dom}\, f$ and let $T : K \to K$ be a QBNE operator with respect to f. If $\mathrm{F}(T)$ is nonempty, then $\{T^n y\}_{n\in\mathbb{N}}$ is bounded for each $y \in K$.

Proof of Theorem 1.4.1

For any $x \in \mathrm{F}(T)$ and $y \in K$ we have

$$D_f(x, T^n y) \leq D_f(x, y)$$

Then the nonnegative sequence $\{D_f(x, T^n y)\}_{n \in \mathbb{N}}$ is bounded. Then

$$f(x) - \langle \nabla f(T^n y), x \rangle + f^*(\nabla f(T^n y)) = D_f(x, T^n y) \leq M$$

This implies that the sequence $\{\nabla f(T^n y)\}_{n \in \mathbb{N}}$ is contained in the sublevel set $\mathrm{lev}_{\leq}^{\psi}(M - f(x))$ of the function $\psi = f^* - \langle \cdot, x \rangle$. Hence, the sequence $\{\nabla f(T^n y)\}_{n \in \mathbb{N}}$ is bounded. Thus the sequence $T^n y = \nabla f^*(\nabla f(T^n y))$, $n \in \mathbb{N}$, is bounded too.

Theorem 1.4.2

Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let K be a nonempty, closed and convex subset of $\mathrm{int}\,\mathrm{dom}\,f$ and let $T : K \to K$ be a BFNE operator with respect to f. If there exists $y \in K$ such that $\|S_n(y)\| \nrightarrow \infty$ as $n \to \infty$, then $\mathrm{F}(T)$ is nonempty.

Definition

For an operator $T : K \to K$, let $S_n(z) := (1/n) \sum_{k=1}^{n} T^k z$ for all $z \in K$.

Some Relevant Implications of our Theorems

Corollary 1.4.3

Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Every nonempty, bounded, closed and convex subset of $\operatorname{int}\operatorname{dom} f$ has the fixed point property for BFNE self-mappings.

Theorem 1.4.4

Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let K be a nonempty, bounded, closed and convex subset of $\operatorname{int}\operatorname{dom} f$. Let $\{T_\alpha\}_{\alpha \in A}$ be a commutative family of BFNE operators with respect to f from K into itself. Then the family $\{T_\alpha\}_{\alpha \in A}$ has a common fixed point.

Theorem 1.5.1

Let $f : X \to (-\infty, +\infty]$ be a Legendre, totally convex function which is positively homogeneous of degree $\alpha > 1$ and uniformly Fréchet differentiable on bounded subsets of X. Let K be a nonempty, bounded, closed and convex subset of $\operatorname{int} \operatorname{dom} f$ with $0 \in K$, and let T be a BFNE self-mapping with respect to f.

Then the following two assertions hold:

(i) For each $t \in (0, 1)$, there exists a unique $u_t \in K$ satisfying $u_t = tTu_t$;

(ii) The net $\{u_t\}_{t \in (0,1)}$ converges strongly to $\operatorname{proj}^f_{\mathrm{F}(T)}(\nabla f^* (0))$ as $t \to 1^-$.

Proof of part (i)

Denote $S_t = tT$ then S_t is a BFNE operator too. Indeed,

$$
\begin{aligned}
\langle \nabla f(S_t x) - \nabla f(S_t y), S_t x - S_t y \rangle &= t^{\alpha} \langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \\
&\leq t^{\alpha} \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \\
&= t^{\alpha - 1} \langle \nabla f(x) - \nabla f(y), S_t x - S_t y \rangle \\
&\leq \langle \nabla f(x) - \nabla f(y), S_t x - S_t y \rangle
\end{aligned}
$$

Since K is bounded, it follows from Corollary 1 that S_t has a fixed point. By the monotonicity of ∇f it follows that the fixed point is unique.

Proof of part (ii)

From Lemma 1 and Theorem 2, $\mathrm{F}(T)$ is nonempty, closed and convex. Thus the Bregman projection $\mathrm{proj}^f_{\mathrm{F}(T)}$ is well defined.

Put $x_n = u_{t_n}$. Since K is bounded, there is a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that $x_{n_k} \rightharpoonup v$. In addition, we have that $\|x_n - Tx_n\| = (t_n - 1)\|Tx_n\|$ and therefore $x_n - Tx_n \to 0$ and hence $v \in \widehat{\mathrm{F}}(T)$. Lemma 2 now implies that $v \in \mathrm{F}(T)$.

Since f is totally convex function, T is a BFNE operator and $Tx_{n_k} \rightharpoonup v$ we get that $Tx_{n_k} \to v$. Now the characterization of the Bregman projection implies that $v = \mathrm{proj}^f_{\mathrm{F}(T)}(\nabla f^*(0))$ and this completes the proof.

Examples of BFNE operators

- **Protoresolvent of A relative to f:**

$$\operatorname{Prt}_A^f := (\nabla f + A)^{-1} : X^* \to 2^X$$

- **Resolvent of A relative to f:**

$$\operatorname{Res}_A^f := \operatorname{Prt}_A^f \circ \nabla f : X \to 2^X$$

- If $A = \partial\varphi$ then we denote:

$$\operatorname{Prox}_\varphi^f := \operatorname{Prt}_{\partial\varphi}^f \text{ and } \operatorname{prox}_\varphi^f := \operatorname{Res}_{\partial\varphi}^f$$

- If K is a nonempty, closed and convex subset of X, then we denote:

$$\operatorname{proj}_K^f := \operatorname{prox}_{\iota_K}^f$$

Examples of BFNE operators

Properties [Bauschke, Borwein and Combettes (2003)]

If $f : X \to \mathbb{R}$ is a cofinite Legendre function and $A : X \to 2^{X^*}$ is maximal monotone mapping, then:

(i) $\operatorname{dom}\left(\operatorname{Res}_A^f\right) = X$;

(ii) Res_A^f is single-valued;

(iii) Res_A^f is a BFNE operator;

(iv) $F(\operatorname{Res}_A^f) = A^{-1}(0^*)$.

Examples of BFNE operators

Conditions

Let K be a nonempty, closed and convex subset of X.

Let $g : K \times K \to \mathbb{R}$ be a bifunction that satisfies the following conditions:

(C1) $g(x, x) = 0$ for all $x \in K$;

(C2) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for all $x, y \in K$;

(C3) for all $x, y, z \in K$,

$$\limsup_{t \downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y);$$

(C4) for each $x \in K$, $g(x, \cdot)$ is convex and lower semicontinuous.

Examples of BFNE operators

The resolvent of bifunction g

$$\operatorname{Res}_g^f(x) = \{z \in K : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in K\}$$

Properties [Reich and Sabach (2010)]

If f is a coercive Legendre function and g satisfies conditions (C1)–(C4), then:

(i) $\operatorname{dom}(\operatorname{Res}_g^f) = X$;

(ii) Res_g^f is single-valued;

(iii) Res_g^f is a BFNE operator;

(iv) $F(\operatorname{Res}_g^f) = EP(g)$.

References

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