Investigations into Normal numbers and Experimental Mathematics

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Definition

A Normal number is an irrational number in which every combination of digits occurs as frequently as any other combination.

That is, when

1. \( a_1 a_2 \cdots a_k \) is any combination of \( k \) digits, and

2. \( N(t) \) is the number of times this combination occurs among the first \( t \) digits in the base \( b \) expansion,

then

\[
\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{b^k}.
\]
## Known Normal Numbers

<table>
<thead>
<tr>
<th>Date</th>
<th>The Number</th>
<th>Author</th>
</tr>
</thead>
<tbody>
<tr>
<td>1933</td>
<td>$0.123456789\ldots$</td>
<td>Champernowne</td>
</tr>
<tr>
<td>1946</td>
<td>$0.23571113\ldots$</td>
<td>Copeland and Erdös Constant</td>
</tr>
<tr>
<td>1952</td>
<td>$0.f(1)f(2)f(3)\ldots$</td>
<td>Davenport and Erdös</td>
</tr>
<tr>
<td>1973</td>
<td>$\sum_{k=1}^{\infty} \frac{1}{b^ek^cck}$</td>
<td>Stoneham</td>
</tr>
<tr>
<td>2001</td>
<td>$0.11011100101_2$</td>
<td>Binary Champerownes, Bailey and Crandall</td>
</tr>
</tbody>
</table>
We investigated the Davenport-Erdös numbers, that when
$f \in \mathbb{Q}[x]$ such that $f(x) \geq 0$ for $x > 0$, have the form
$0.f(1)f(2)f(3)\ldots$.
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0.f(1)f(2)f(3)\ldots
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Theorem (?)

Let \( f(x) \in \mathbb{Q}[x] \), so that when \( x \in \mathbb{N} \), \( f(x) \geq 0 \). Then the decimal \( .f(1)f(2)f(3)\ldots_{10} \) is 10-normal.
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Definition (Simply Strongly Normal)

1. Let $\alpha \in \mathbb{R}$ with base-$b$ fractional part $0.a_0a_1a_2\ldots$, and

2. $m_k(n) := \# \{ i : a_i = k, i \leq n \}$.

$\alpha$ is simply strongly normal in base $b$ if for each $0 \leq k \leq b - 1$

$$
\limsup_{n \to \infty} \frac{m_k(n) - n/b}{\sqrt{2n \log \log n}} = \frac{\sqrt{b-1}}{b}, \text{ and}
$$

$$
\liminf_{n \to \infty} \frac{m_k(n) - n/b}{\sqrt{2n \log \log n}} = -\frac{\sqrt{b-1}}{b}.
$$
Moreover...

A number is strongly normal in base $b$ if it is simply strongly normal in each base $b^j$ for $j = 1, 2, 3, \ldots$, and is absolutely strongly normal if it is strongly normal in every base.

It was proved in (?)

1. If a number is strongly normal, it is normal.

2. “Almost all” numbers are strongly normal in any base.
Strong Normality

Let $\alpha \in \mathbb{R}$ have base ten expansion $0.a_1a_2\ldots$, and take

$$p_k(n) = \frac{m_k(n) - n/b}{\sqrt{2n \log \log n}}$$

(this is equivalent to removing the limits from the definition of Simply Strongly Normal).

We plot, for various Davenport-Erdős numbers, $p$ against $n$ for all $k$ of $\alpha$ and can observe whether $p_k(n)$ is tending towards $\pm \frac{3}{10}$ or not.
## Strong Normality

<table>
<thead>
<tr>
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<th>k</th>
</tr>
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<td>Green</td>
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<tr>
<td>Blue</td>
<td>2</td>
</tr>
<tr>
<td>Coral</td>
<td>3</td>
</tr>
<tr>
<td>Orange</td>
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<table>
<thead>
<tr>
<th>Colour</th>
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<tr>
<td>Turquoise</td>
<td>8</td>
</tr>
<tr>
<td>Maroon</td>
<td>9</td>
</tr>
</tbody>
</table>
\( f(x) = x \) for \( n = 1, \ldots, 10^6 \).
\( f(x) = x \) for \( n = 1, \ldots, 10^7 \).
\[ f(x) = x \text{ for } n = 1, \ldots, 10^8. \]
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\[ f(x) = x^2 \] for \( n = 1, \ldots, 10^7 \).
\[ f(x) = x^2 \text{ for } n = 1, \ldots, 10^8. \]
$f(x) = x^3$ for $n = 1, \ldots, 10^6$. 
\[ f(x) = x^3 \text{ for } n = 1, \ldots, 10^7. \]
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\[ f(x) = 3x^3 - 2x^2 + x \text{ for } n = 1, \ldots, 10^7. \]
$f(x) = 3x^3 - 2x^2 + x$ for $n = 1, \ldots, 10^8$. 
Conjecture

Let \( f(x) \in \mathbb{Q}[x] \), then the decimal

\[ .f(1)f(2)f(3) \ldots_{10} \]

is not strongly normal when \( x \in \mathbb{N} \) and \( f(x) \geq 0 \).
Strong Normality

We repeated these graphs using famous (possibly normal?) constants rather than Davenport-Erdös numbers and observed something interesting...
Strong Normality

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We shall see that with these constants $p_k(n)$ is generally bound between $\pm \frac{3}{10}$ as $n \to \infty$. 
\[\pi \text{ for } n = 1, \ldots, 10^6\]
\[ \pi \text{ for } n = 1, \ldots, 10^7 \]
$\pi$ for $n = 1, \ldots, 10^8$
$e$ for $n = 1, \ldots, 10^6$
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\( \varphi = \frac{1+\sqrt{5}}{2} \) for \( n = 1, \ldots, 10^6 \)
φ = \frac{1+\sqrt{5}}{2} \text{ for } n = 1, \ldots, 10^7
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\log(2) \text{ for } n = 1, \ldots, 10^6
$\log(2)$ for $n = 1, \ldots, 10^7$
\[ \log(2) \text{ for } n = 1, \ldots, 10^8 \]
Catalan’s Constant for $n = 1, \ldots, 10^6$
Catalan’s Constant for $n = 1, \ldots, 10^7$
Catalan’s Constant for $n = 1, \ldots, 10^8$
\( \zeta(3) \) for \( n = 1, \ldots, 10^6 \)
\[ \zeta(3) \text{ for } n = 1, \ldots, 10^7 \]
$\zeta(3)$ for $n = 1, \ldots, 10^8$
Strong Normality

Conjecture

$\pi, \zeta(3), e, \log(2), \varphi$ and Catalan’s Constants constant are simply strongly normal in base 10.
Future Work

1. Plots for more digits, $10^{10}+$.

2. Plots in other bases apart from 10.

3. Comparing $p_k(n)$ with other functions.

4. A proof for the above conjectures.

5. A proof of the irrationality of $\pi + e$ (Probably above my calibre at this point).

6. Investigations into big data.
Acknowledgements

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▶ Mr Matthew Tam,

▶ Dr Paul Vbrik,

▶ Mr Corey Sinnamon,

▶ Mr Ghislain McKay,

▶ Mr Tony Jackson, and

▶ and all the wonderful people in the Mathematics department.

Thank you!