Solving Partial Differential Equations with Finite Element Methods

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"The art of doing mathematics consists in finding that special case which contains all the germs of generality" by D. Hilbert (1862-1943).
1. An Image Processing Problem
2. Plate Deformation and Stokes Problem
Noisy Signal

An Image Processing Problem

- When a signal or an image is transmitted over some faulty communication lines, they become corrupted.
- The exact nature of corruption is not known a priori. However, the corruption can be modelled as Gaussian or impulsive noise or the mixture of both.

- Only Gaussian noise ⇒ Use Fourier or Wavelets smoothing
- Only impulsive ⇒ Detect the impulses, remove them and reconstruct using interpolation
- Mixture of both ⇒ Detect the impulses, remove them and interpolate but how to filter the Gaussian noise? Fourier and Wavelets perform badly on smoothing scattered data

Let $\mathcal{T}$ be the operator modelling the faulty transmission line. Modelling the signal as a vector $\mathbf{f}$, the new signal is $\mathbf{z} = \mathcal{T}\mathbf{f}$ or $z_i = \mathcal{T}f_i$ component-wise.
Use the finite element smoothing for the mixture noise. The basic idea is to take into account the derivative of the signal. Here $\{z_i\}_{i=1}^N$ is the corrupted signal but $u(x, y)$ is the recovered signal. We want to solve the minimisation problem with $V$ as the finite element space:

$$
\min_{u \in V} \sum_{i=0}^{N} (u(x_i, y_i) - z_i)^2 + \lambda \int_{\Omega} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dx \, dy
$$

control the error

control the smoothness
For a prescribed body force $f \in [L^2(\Omega)]^d$, the Stokes equations with homogeneous Dirichlet boundary condition on the boundary of $\Omega$ reads

$$
-\nu \Delta u + \nabla p = f \quad \text{in} \quad \Omega
$$

$$
\text{div} \, u = 0 \quad \text{in} \quad \Omega
$$

with $\mathbf{u} = \mathbf{0}$ on the boundary of $\Omega$. where $\mathbf{u}$ is the velocity, $p$ is the pressure, and $\nu$ denotes the viscosity of the fluid.
Mixed formulation of Reissner–Mindlin plate with clamped boundary condition is to find $(\phi, u, \zeta) \in [H^1_0(\Omega)]^2 \times H^1_0(\Omega) \times [L^2(\Omega)]^2$ such that

$$\int_{\Omega} C \epsilon(\phi) : \epsilon(\psi) \, dx + \int_{\Omega} (\psi - \nabla v) \cdot \zeta \, dx = \ell(v), \quad (\psi, v) \in [H^1_0(\Omega)]^2 \times H^1_0(\Omega),$$

$$\int_{\Omega} (\phi - \nabla v) \cdot \beta \, dx - \frac{t^2}{\lambda(1 - t^2)} (\zeta, \beta) = 0, \quad \beta \in [L^2(\Omega)]^2$$

$u$: transverse displacement

$\phi$: rotation of the transverse normal vector

$\zeta$: shear stress (Lagrange multiplier)
Thank You!!!

Thank You