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Borwein, Jonathan M. (3-DLHS-CDC); **Vanderwerff, Jon D.**
★**Convex functions: constructions, characterizations and counterexamples.**
Encyclopedia of Mathematics and its Applications, 109.
Cambridge University Press, Cambridge, 2010. *x*+521 *pp*. $130.00. *ISBN* 978-0-521-85005-6

The systematic study of convex functions—and, more broadly, convex analysis and optimization—can be traced back to a set of lecture notes by W. Fenchel from 1951. The theory was greatly expanded upon in the 1960s in many works by J.-J. Moreau and by R. T. Rockafellar, and it continues to be a very active area of research today.

In the 521-page book under review, Borwein and Vanderwerff aim their focus on convex functions themselves. The book is organized into ten chapters, with extensive chapter notes and numerous exercises. A brief overview follows.

Chapter 1 is motivational and illustrates the ubiquity of convexity by many examples and results. The reader will find basic results on the generation of convex functions, related concepts such as subgradients, a list of classical theorems due to Radon, Helly, Carathéodory, etc., and basic calculus rules. Borwein and Vanderwerff make the connection to the Chebyshev problem, the calculus of variations (via the brachistochrone problem), Aumann and Lyapunov convexity theorems, and various other delightful examples and constructions.

The fundamental Euclidean case is developed in detail in Chapter 2. Most of the results (on (sub-)differentiability, continuity, and Fenchel conjugation and duality) in this chapter are classical and can be found, e.g., in the books by A. W. Roberts and D. E. Varberg [*Convex functions*, Academic Press, New York, 1973; MR0442824 (56 #1201)] and by R. T. Rockafellar [*Convex analysis*, Princeton Mathematical Series, No. 28, Princeton Univ. Press, Princeton, NJ, 1970; MR0274683 (43 #445)]. A distinctive feature is a complete presentation of the deep differentiability results due to Rademacher and to Alexandrov.

The third chapter focuses on more advanced topics in Euclidean spaces such as polyhedral convex functions, functions of eigenvalues, duality for linear and semidefinite optimization, and the Kakutani-Fan fixed point theorem.

With Chapter 4, the authors leave the Euclidean setting and enter the realm of general Banach spaces. Accordingly, the basic theory on various continuities, differentiabilities, Fenchel conjugation, and corresponding calculus is developed. The authors also touch upon the Chebyshev problem and variational principles.

Starting off with norms, Borwein and Vanderwerff provide in Chapter 5 a detailed study on the duality between various strict convexity notions and the corresponding smoothness notions.

Miscellaneous topics are covered in the sixth chapter, including finer properties of the subdifferential operator, selection theorems, various notions of convergence for functions and sets, convex integral functions, strongly rotund functions, trace class spectral functions, Asplund spaces, the Radon-Nikodým property, and convex functions on lattices.

An up-to-date account on barriers and Legendre functions in reflexive and nonreflexive Banach spaces is presented in Chapter 7. These functions have proven to be useful in projection and similar methods that are based on the Bregman distance as a measure of discrepancy between points.

The coincidence of differentiability notions gives rise to various characterizations of Banach spaces. This deep structure theory is developed in Chapter 8; here is a typical result provided by the authors. For a Banach space X and its dual X^*, the following are equivalent: (i) X is finite-dimensional; (ii) weak* and norm sequential convergence coincide in X^*; (iii) every continuous convex function on X is bounded on bounded sets; (iv) every continuous convex Gateaux differentiable function on X is Fréchet differentiable.

Chapter 9 contains the basic results on maximal monotone operators. The authors develop the theory by using the (convex) Fitzpatrick function, which has had a profound impact on the still evolving theory of maximal monotone operators. Noteworthy is a thorough account presented on the additive Asplund decomposition of a maximal monotone operator into a subdifferential part and an acyclic part.

The authors conclude their journey with Chapter 10, where they reflect back on the results of the previous chapters and where they present characterizations of finite-dimensional Banach spaces within the class of Banach spaces.

Borwein and Vanderwerff's book is particularly impressive due to its enormous breadth and depth. Necessarily, not all results are developed in complete detail—in fact, many proofs are left as exercises—which makes the text less ideal as a reference work; however, complete details would have easily at least doubled the number of pages. Some of the results are revisited, especially when simpler proofs are available in more restrictive settings. It is a beautiful experience to browse this inspiring book. The reviewer has not seen any source which is even close to presenting so many different and interesting convex functions and corresponding results. The book does contain typos but the authors provide a very detailed and complete list of errata, additional notes, and even solutions to many exercises on their website.

In summary, this delightful book is a most welcome addition to the library of any convex analyst or of any mathematician with an interest in convex functions.

Reviewed by *Heinz H. Bauschke*