of $\pi$. Also, as we will see in the next section, there do exist formulas for certain other constants that admit individual digit calculation schemes in various nonbinary bases (including base ten).

3.5 Unpacking the BBP Formula for Pi

It is worth asking “why” the formula

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$  \hspace{1cm} (3.38)

exists. As observed above, this identity is equivalent to, and can be proved by establishing:

$$\pi = \int_{0}^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8} dx.$$  \hspace{1cm} (3.39)

The present version of Maple evaluates this integral to

$$-2\log 2 + 2\log(2 - \sqrt{2}) + \pi + 2\log(2 + \sqrt{2}),$$

which simplifies to $\pi$. In any event, one can ask what the individual series in (3.38) comprise. So consider

$$S_b = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+b)}$$

Table 3.4. Computed hexadecimal digits of $\pi$. 

<table>
<thead>
<tr>
<th>Position</th>
<th>Hex Digits Beginning at This Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^6$</td>
<td>26C65E52CB4593</td>
</tr>
<tr>
<td>$10^7$</td>
<td>17AF5863EFED8D</td>
</tr>
<tr>
<td>$10^8$</td>
<td>ECB840E21926EC</td>
</tr>
<tr>
<td>$10^9$</td>
<td>85895585A0426B</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>921C73C6838FB2</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>9C381872D27596</td>
</tr>
<tr>
<td>$1.25 \times 10^{12}$</td>
<td>07E45733CC790B</td>
</tr>
<tr>
<td>$2.5 \times 10^{14}$</td>
<td>E6216B069CB6C1</td>
</tr>
</tbody>
</table>
for $1 \leq b \leq 8$ and the corresponding normalized integrals
\[
I(b) = 2^{b/2} \int_0^{1/\sqrt{2}} x^{b-1} \frac{1}{1-x^8} \, dx. \tag{3.40}
\]
Again, Maple provides closed forms for $I(b)$ in which the basic quantities seem to be the following: $\arctan(2)$, $\arctan(1/2)$, $\sqrt{2} \arctan(1/\sqrt{2})$, $\log(2)$, $\log(3)$, $\log(5)$, and $\log(\sqrt{2} \pm 1)$. At this point one may use integer relation methods and obtain:
\[
S_1 = \frac{\pi}{8} + \log \frac{5}{8} - \sqrt{2} \log(\sqrt{2} - 1) - \frac{\arctan(1/2)}{4} + \frac{\sqrt{2} \arctan(\sqrt{2}/2)}{4}
\]
\[
S_2 = \log \frac{3}{4} + \frac{\arctan(1/2)}{2}
\]
\[
S_3 = \frac{\pi}{4} - \frac{\sqrt{2} \log(\sqrt{2} - 1)}{2} - \arctan(1/2) - \frac{\sqrt{2} \arctan(\sqrt{2}/2)}{2} - \frac{\log 5}{4}
\]
\[
S_4 = \frac{\log 5}{2} - \frac{\log 3}{2}
\]
\[
S_5 = -\frac{\pi}{2} - \sqrt{2} \log(\sqrt{2} - 1) + \arctan(1/2) + \sqrt{2} \arctan(\sqrt{2}/2) - \frac{\log 5}{2}
\]
\[
S_6 = \log 3 - 2 \arctan(1/2)
\]
\[
S_7 = -\pi + \log 5 - 2 \sqrt{2} \log(\sqrt{2} - 1) + 2 \arctan(1/2) - 2 \sqrt{2} \arctan(\sqrt{2}/2)
\]
\[
S_8 = 8 \log 2 - 2 \log 5 - 2 \log 3. \tag{3.41}
\]
Thus the “simple” hexadecimal formula (3.38) is actually a molecule made up of more subtle hexadecimal atoms: with the final bond coming from the simple identity $\arctan 2 + \arctan(1/2) = \pi/2$. As an immediate consequence, one obtains the formula $\arctan(1/2) = S_2 - S_6/4$.

Furthermore, the facts that
\[
\text{Im} \left( \log \left( 1 - \frac{1-i}{x} \right) \right) = \arctan \left( \frac{1}{1-x} \right)
\]
\[
2 \arctan(1/3) + \arctan(1/7) = \arctan(1/2) + \arctan(1/3) = \arctan 1 = \pi/4 \tag{3.42}
\]
allow one to write directly a base-64 series for $\arctan(1/3)$ (using $x = 4$) and a base-1024 series for $\arctan(1/7)$ (using $x = 8$). This yields the identity
\[
\frac{\pi}{4} = \frac{1}{16} \sum_{n=0}^{\infty} \frac{(-1)^n}{64^n} \left( \frac{8}{4n+1} + \frac{4}{4n+2} + \frac{1}{4n+3} \right) + \frac{1}{256} \sum_{n=0}^{\infty} \frac{(-1)^n}{1024^n} \left( \frac{32}{4n+1} + \frac{8}{4n+2} + \frac{1}{4n+3} \right), \tag{3.43}
\]
which is similar to, although distinct from, the identity used by Bellard and Percival in their computations.

3.6 Other BBP-Type Formulas

A formula of the type mentioned in the previous sections, namely

\[ \alpha = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}, \quad (3.44) \]

is now referred to as a BBP-type formula, named after the initials of the authors of the 1997 paper where the \( \pi \) hex digit algorithm appeared [21]. For a constant \( \alpha \) given by a formula of this type, it is clear that individual base-\( b \) digits can be calculated, using the scheme similar to the ones outlined in the previous section. The paper [21] includes formulas of this type for several other constants. Since then, a large number of other BBP-type formulas have been discovered.

Most of these identities were discovered using an experimental approach, using PSLQ searches. Others were found as the result of educated guesses based on experimentally obtained results. In each case, these formulas have been formally established, although the proofs are not always as simple as the proof of Theorem 3.1. We present these results, in part, to underscore the fact that the approach used to find the new formula for \( \pi \) has very broad applicability.

A sampling of the known binary BBP-type formulas (i.e., formulas with a base \( b = 2^p \) for some integer \( p \)) is shown in Table 3.5. Some nonbinary BBP-type formulas are shown in Table 3.6. These formulas are derived from several sources: [21, 60, 61]. An updated collection is available at [16]. The constant \( G \) that appears in Table 3.5 is Catalan’s constant, namely \( G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \approx 0.9159655941\ldots \)

In addition to the formulas in Tables 3.5 and 3.6, there are two other classes of constants known to possess binary BBP-type formulas. The first is logarithms of certain integers. Clearly, \( \log n \) can be written with a binary BBP formula (i.e. a formula with \( b = 2^m \) for some integer \( m \)) provided \( n \) factors completely using primes whose logarithms have binary BBP formulas—one merely combines the individual series for the different primes into a single binary BBP formula. We have seen that the logarithm of the prime 2 possesses a binary BBP formula, and so does \( \log 3 \), by the following reasoning: