31. E. Landau, Ein Satz über die Zerlegung homogener linearer Differentialausdrücke in irreduzible Faktoren, J. Reine Angew. Math., 124 (1902) 115–120.

32. S. Lang, Introduction to algebraic geometry, Interscience, New York, 1958.

33. A. Loewy, Über reduzible homogene Differentialausdrücke, Math. Ann., 56 (1903) 549–584.

34. M. Nagata, A remark on the unique factorization theorem, J. Math. Soc. Japan, 9 (1957) 143–145.

35. ———, Local rings, Interscience, New York — London, 1962.

36. P. Samuel, Sur les anneaux factoriels, Bull. Soc. Math. France, 89 (1961) 155–178.

37. ———, Anneaux factoriels, Sao Paulo, 1963.

38. ———, Lectures on unique factorization domains, TIFR, Bombay, 1964.

39. ———, Unique factorization, this MONTHLY, 75 (1968) 945–952.

40. S. Stevin, Arithmétique (1585, new ed. 1958).

41. B. L. van der Waerden, Moderne algebraische Geometrie, Springer, Berlin, 1939.

42. O. Zariski and P. Samuel, Commutative algebra I, Van Nostrand, Princeton, 1968.

CONTINUOUS ANALOGUES OF SERIES

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1. Introduction. The present note, which was inspired by [10], arose as an attempt to understand why some infinite series have continuous analogues whereas others do not. The methods of [10] are *ad hoc* and fail to reveal the underlying mechanism.

The notion of continuous analogue is not easy to define precisely; however, given that [1], [6], [7]

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2(c+n)\alpha}{(c+n)^2} = \int_{-\infty}^{\infty} \frac{\sin^2(c+x)\alpha}{(c+x)^2}\,dx = \pi/\alpha, \qquad 0<\alpha<\pi, \tag{1}$$

where $(\sin^2 \alpha u)/u^2$ is taken as α^2 when $u = 0$, almost anyone would call the integral in (1) a continuous analogue of the series. A less transparent example is (series: see, for example, [5], p. 102; integral: see any book on complex analysis)

$$\sum_{n=1}^{\infty} \frac{\sin(n-\frac{1}{2})\alpha}{n-\frac{1}{2}} = \int_{0}^{\infty} \frac{\sin \alpha x}{x}\,dx = \frac{\pi}{2}\operatorname{sgn}\alpha, \qquad |\alpha| < 2\pi, \tag{2}$$

where the analogy seems flawed; but it is improved if we rewrite (2) as

$$\sum_{n=-\infty}^{\infty} \frac{\sin(n-\frac{1}{2})\alpha}{n-\frac{1}{2}} = \int_{-\infty}^{\infty} \frac{\sin(x-\frac{1}{2})\alpha}{x-\frac{1}{2}}\,dx. \tag{3}$$

Recently Pollard and Shisha [10] observed that although the binomial series

$$(1+e^{it})^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}e^{int}, \quad |t|<\pi, \ -\pi<t<\pi; \ \alpha>-1 \tag{4}$$

does not have a continuous analogue — that is, we do not get a correct result by replacing n in (4) by x and replacing summation by integration — it does if we first extend the sum in (4) over $(-\infty,\infty)$. This does not change the series because the added terms are all zero, and it turns out that, in fact,

$$\int_{-\infty}^{\infty}\binom{\alpha}{u}e^{iut}du = \sum_{n=-\infty}^{\infty}\binom{\alpha}{n}e^{int} = (1+e^{it})^\alpha \tag{5}$$

for $\alpha>-1$ and $|t|<\pi$, $-\pi<t<\pi$.

Thus a series that does not have a continuous analogue may acquire one if we write a different but equivalent formula for the terms of the series.

As another example, it originally puzzled us that the binomial series (4) has a continuous analogue whereas the equally natural

$$(1-e^{it})^{-\alpha} = \sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}e^{int}, \qquad \alpha<1, \ 0<t<2\pi \tag{6}$$

does not, even if we extend the sum over $(-\infty,\infty)$.

If, however, we write (6) in the equivalent form

$$(1-e^{it})^{-\alpha} = \sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}\frac{\sin\pi(n+\alpha)}{\sin\pi\alpha}e^{in\pi}e^{int}, \qquad 0<t<2\pi, \tag{7}$$

it is true (as we shall see) that also

$$(1-e^{it})^{-\alpha} = \int_{-\infty}^{\infty}\binom{\alpha+u-1}{u}\frac{\sin\pi(u+\alpha)}{\sin\pi\alpha}e^{iu\pi}e^{iut}du. \tag{8}$$

Continuous analogues of series are of interest in physics (cf. [**2**]), where one often attempts to deal with an intractable sum by replacing it by the corresponding integral. In fact, the sum in (1) does arise in physics and was "approximated" by the integral before it was realized that the approximation is exact. (See [**1**], [**6**], [**7**].)

2. A general formula. We shall need the notation, but only the simplest theorems, of the theory of Fourier transforms. (Everything that we use is in [**11**].) Our notation is

$$g(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-iux}G(u)du, \quad G(u) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}g(x)e^{iux}dx, \tag{9}$$

and we shall use other letters in the same way. In all our work, G will be zero outside a certain finite interval, and we shall suppose that $|G|$ is integrable.

Suppose now that $G(x)=0$ for x outside an interval $(t-2\pi, t+2\pi)$, with $G(x)\to 0$ as $x\to t\pm 2\pi$ from inside the interval. Then when n is an integer,

$$(2\pi)^{\frac{1}{2}}g(n) = \int_{t-2\pi}^{t} e^{-inu}G(u)du + \int_{t}^{t+2\pi} e^{-inu}G(u)du$$

$$= \int_{t}^{t+2\pi} e^{-inu}\{G(u-2\pi)+G(u)\}du.$$

The numbers on the right are 2π times the Fourier coefficients for the interval $(t, t+2\pi)$ of the function $G(u-2\pi)+G(u)$. Let us suppose that $G(u-2\pi)+G(u)$ satisfies, at t, a sufficient condition for the convergence of its Fourier series to its value at t (for example, it is enough to have the function of bounded variation in a neighborhood of t, its value at t being the average of its right-hand and left-hand limits). Then $\sum g(n)e^{int}$ is a Fourier series and converges to the value at t of the function that generates it; that is,

$$\sum_{n=-\infty}^{\infty} g(n)e^{int} = (2\pi)^{\frac{1}{2}}\{G(t-2\pi)+G(t)\} = (2\pi)^{\frac{1}{2}}G(t).$$

If we replace $G(t)$ by its value from (9), we find

$$\sum_{n=-\infty}^{\infty} g(n)e^{int} = \int_{-\infty}^{\infty} g(x)e^{itx}dx. \tag{10}$$

In practice we usually have $G(x) = 0$ outside a shorter interval (r, s); then (10) holds for $s - 2\pi < t < r + 2\pi$. In particular, it holds for $0 < t < 2\pi$ when $(r, s) = (0, 2\pi)$; and for $-\pi < t < \pi$ when $(r, s) = (-\pi, \pi)$.

Formula (10) is actually a special case of the Poisson summation formula ([**11**], p. 60; [**13**], vol. I, p. 68; **2**; [**8**], p. 152), but we do not need the general formula.

3. Examples. We can now produce examples of (10) by looking at functions that are known to have the form

$$g(x) = (2\pi)^{-\frac{1}{2}} \int_{r}^{s} e^{-iux}G(u)du, \qquad s - r < 4\pi \tag{11}$$

(cf. [**1**], [**4**], [**6**], [**7**]).

Let us first take $G(u) = e^{-icu}$ on $(-\alpha, \alpha)$ and $G(u) = 0$ for $|u| > \alpha$, where $0 < \alpha < \pi$. Then

$$g(x) = (2\pi)^{-\frac{1}{2}} \int_{-\alpha}^{\alpha} e^{-icu-ixu}du = 2(2\pi)^{\frac{1}{2}} \frac{\sin(c+x)\alpha}{c+x}, \tag{12}$$

with the convention that $u^{-1}\sin\alpha u = \alpha$ when $u = 0$. Hence we have (10) for $\alpha - 2\pi < t < \alpha + 2\pi$, that is

$$\sum_{n=-\infty}^{\infty} \frac{\sin(c+n)\alpha}{c+n} e^{int} = \int_{-\infty}^{\infty} \frac{\sin(c+x)\alpha}{c+x} e^{itx}dx,$$

provided that $0 < \alpha < \pi$, $-\pi < t < 2\pi$. In particular, we can take $t = 0$ and then

$$\sum_{n=-\infty}^{\infty} \frac{\sin(c+n)\alpha}{c+n} = \int_{-\infty}^{\infty} \frac{\sin(c+x)\alpha}{c+x} dx = \pi;$$

since the integrand is an odd function the formula can be written in the form

$$\sum_{n=-\infty}^{\infty} \frac{\sin(c+n)\alpha}{c+n} = \int_{-\infty}^{\infty} \frac{\sin(c+x)\alpha}{c+x} dx = \pi \operatorname{sgn} \alpha, \qquad |\alpha| < \pi.$$

For $c = \frac{1}{2}$ we have (2) and for $c = 0$ we have

$$\sum_{n=-\infty}^{\infty} \frac{\sin n\alpha}{n} = \int_{-\infty}^{\infty} \frac{\sin \alpha x}{x} dx = \pi \operatorname{sgn} \alpha, \qquad |\alpha| < \pi;$$

remembering that the term with $n = 0$ is to be interpreted as α, we have a symmetrical version of a familiar Fourier expansion.

This discussion can be generalized. The product of two Fourier transforms g_1 and g_2 is again a Fourier transform; if G_1 and G_2 vanish outside $(-\alpha, \alpha)$ and $(-\beta, \beta)$, respectively, then $g_1 g_2$ is the transform of a function vanishing outside $(-\alpha - \beta, \alpha + \beta)$ (actually the function for which $g_1 g_2$ is the transform is the convolution of G_1 and G_2, but we do not need to know this). Hence if

$$g_1(x) = (2\pi)^{-\frac{1}{2}} \int_{-\alpha}^{\alpha} e^{-iux} G_1(u) du \quad \text{and} \quad g_2(x) = (2\pi)^{-\frac{1}{2}} \int_{-\beta}^{\beta} e^{-iux} G_2(u) du,$$

and $\alpha + \beta < 2\pi$, then

$$\sum_{n=-\infty}^{\infty} g_1(n) g_2(n) e^{int} = \int_{-\infty}^{\infty} g_1(x) g_2(x) e^{itx} dx$$

provided that $\alpha + \beta - 2\pi < t < -\alpha - \beta + 2\pi$.

Taking g_1 as in (12) and $t = 0$, we have in particular

$$\sum_{n=-\infty}^{\infty} g_2(n) \frac{\sin(c+n)\alpha}{c+n} = \int_{-\infty}^{\infty} g_2(x) \frac{\sin(c+x)\alpha}{c+x} dx. \tag{13}$$

If we further specialize by taking g_2 equal to g_1, we get

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2(c+n)\alpha}{(c+n)^2} = \int_{-\infty}^{\infty} \frac{\sin^2(c+x)\alpha}{(c+x)^2} dx = \frac{\pi}{\alpha}, \qquad 0 < \alpha < \pi; \tag{14}$$

this is formula (1) of §1.

4. Binomial series. With Pollard and Shisha, we start from the formula

$$\binom{\alpha}{x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixt}(1 + e^{it})^{\alpha} dt, \qquad \alpha > -1, \ -\infty < x < \infty. \tag{15}$$

This is of the form (11) with $r = -\pi$, $s = \pi$; the Fourier series of $(1 + e^{t})^{\alpha}$ converges for $|t| < \pi$ by almost any convergence test that we might think of applying.

Then (10) takes the form (5).

Pollard and Shisha also give

$$(16)\qquad \sum_{n=-\infty}^{\infty}\binom{\alpha}{n+c}e^{i(n+c)t}=\int_{-\infty}^{\infty}\binom{\alpha}{u+c}e^{i(u+c)t}du=(1+e^{it})^{\alpha},$$

under the same conditions as (5); this is (10) again with

$$G(t)=\begin{cases}e^{-ict}\,(1+e^{it})^{\alpha}, & |t|<\pi;\\ 0, & |t|>\pi.\end{cases}$$

The pair (7), (8) follow in the same way, once we realize that (15) can be transformed into

$$e^{-i\pi x}\,\frac{\sin\pi(\alpha+x)}{\sin\pi\alpha}\binom{\alpha+x-1}{x}=\frac{1}{2\pi}\int_0^{2\pi}(1-e^{it})^{-\alpha}e^{-itx}dt$$

by the usual formulas about the gamma function.

5. A general method. The preceding discussion suggests a general method for constructing continuous analogues of series. Consider a function f defined on the integers, and let $\sum_{n=-\infty}^{\infty}f(n)e^{int}$ be the Fourier series of an integrable function F, so that

$$f(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inu}F(u)du.$$

There are many conditions that are sufficient for this, for example that $\sum|f(n)|$ converges or that $f(n)\to 0$ and f is even and convex ([**13**], vol. I, pp. 183, 326). The function ϕ defined for all real x by

$$\phi(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ixu}\,F(u)du$$

interpolates f at the integers and has the form (11). Consequently we have

$$\sum_{n=-\infty}^{\infty}f(n)e^{int}=\sum_{n=-\infty}^{\infty}\phi(n)e^{int}=\int_{-\infty}^{\infty}\phi(x)e^{ixt}dx,\qquad |t|<\pi.$$

That is, we can construct a continuous analogue of any Fourier series that belongs to a function satisfying the conditions imposed on G in §2. Whether we are willing to regard this as a reasonable analogue seems to depend on whether we can write $\phi(x)$ in a sufficiently recognizable form.

Let us see how the method works out in some specific examples.

We first look for a continuous analogue of the logarithmic series, which we can write in the form

$$it=\sum_{n\neq 0}\frac{(-1)^{n-1}}{n}e^{int},\qquad |t|<\pi.$$

Here

$$f(n) = \frac{i}{2\pi}\int_{-\pi}^{\pi} ue^{-inu}du, \qquad n \neq 0;\ f(0) = 0;$$

$$F(u) = iu,$$

$$\phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixu}F(u)du = \frac{1}{x}\left(\cos\pi x - \frac{\sin\pi x}{x}\right).$$

It is clear that $\phi(n)$ does in fact equal $f(n)$, although there is no really natural compelling analogue of $(-1)^{n-1}/n$, and $\phi(x)$ is perhaps not the most obvious interpolating function. Our continuous analogue is

$$\sum_{n\neq 0} \frac{(-1)^{n-1}}{n}e^{int} = \sum_{-\infty}^{\infty} \frac{1}{n}\left(\cos n\pi - \frac{\sin n\pi}{n\pi}\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{x}\left(\cos\pi x - \frac{\sin\pi x}{\pi x}\right)e^{ixt}dx = it,\ |t| < \pi.$$

This seems acceptable; indeed, ϕ is the only possible interpolating function of the form (11).

Now let us look for a continuous analogue of the exponential series,

$$\sum_{n=0}^{\infty} \frac{1}{n!}e^{int} = \exp(e^{it}).$$

Here

$$\phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixu}\exp(e^{iu})du, \tag{17}$$

and we have $\phi(n) = 1/n!$, $n = 0,1,2,\cdots$; $\phi(n) = 0$, otherwise;

$$\sum_{n=0}^{\infty} \frac{1}{n!}e^{int} = \int_{-\infty}^{\infty} e^{ixt}dx\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixu}\exp(e^{iu})du.$$

This, although formally a pair of analogues, seems unsatisfactory, partly at least because we are conditioned to expect a continuous analogue of $1/n!$ to involve $1/\Gamma(x+1)$, whereas $\phi(x) = 1/\Gamma(x+1)$ only when $x = n$.

We note that a function $\phi(x)$ of the form (18) is easily seen to be (a) bounded on the real axis, (b) of exponential type in the plane, i.e., $|\phi(z)| \leqq Ae^{\pi|z|}$. But $1/\Gamma(x+1)$ does not satisfy either (a) or (b), for example because $1/\Gamma(-n+\frac{1}{2}+1) = (-1)^{n-1}\Gamma(n-\frac{1}{2})\pi \to \infty$ faster than any e^{Bn} (by Stirling's formula).

We should accordingly like to put $\phi(x)$ into a form that involves $1/\Gamma(x+1)$ explicitly. Now a well-known formula (see [3], vol. 1, p. 13) states that

$$\frac{1}{\Gamma(z+1)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-z-1}e^t dt,$$

where the path of integration can be taken to be the loop extending from $-\infty$ to -1 along the real axis, around zero on the unit circumference, and back to $-\infty$. This yields

$$\phi(x) = \frac{1}{\Gamma(1+x)} - \frac{\sin \pi x}{\pi} \int_1^\infty \frac{e^{-u}}{u^{1+x}} du,$$

and consequently with this $\phi(x)$

$$\sum_{n=0}^{\infty} \frac{1}{n!} e^{int} = \sum_{-\infty}^{\infty} \phi(n) e^{int} = \int_{-\infty}^{\infty} \phi(x) e^{ixt} dx.$$

5. Bessel functions. The generating function for the Bessel functions $J_n(s)$ is

$$\exp\left(\tfrac{1}{2}s(w - 1/w)\right) = \sum_{n=-\infty}^{\infty} w^n J_n(s),$$

or, with $w = e^{it}$,

$$\exp\left(\tfrac{1}{2}s(e^{it} - e^{-it})\right) = \sum_{n=-\infty}^{\infty} e^{int} J_n(s). \tag{18}$$

Let us look for a continuous analogue of (18). Bessel's integral ([**12**], p. 19) is

$$J_n(s) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - s \sin \theta) d\theta,$$

when n is an integer. Replacing n by x yields a function of the form (11); unfortunately it is not $J_x(s)$ when x is not an integer, but is known as Anger's function $\mathbf{J}_x(s)$ ([**12**], p. 308). What is true is that ([**12**], p. 176)

$$J_x(s) + h_x(s) = \frac{1}{\pi} \int_0^\pi \cos(x\theta - s \sin \theta) d\theta,$$

where

$$h_x(s) = \frac{\sin \pi x}{\pi} \int_0^\infty e^{-xr - s \sinh r} dr = \mathbf{J}_x(s) - J_x(s),$$

and $h_x(s) = 0$ when x is an integer. Hence an analogue of (18) is

$$\sum_{-\infty}^{\infty} e^{int}(J_n(s) + h_n(s)) = \int_{-\infty}^{\infty} e^{ixt}(J_x(s) + h_x(s)) dx,$$

or alternatively

$$\sum_{-\infty}^{\infty} e^{int} J_n(s) = \sum_{-\infty}^{\infty} e^{int} \mathbf{J}_n(s) = \int_{-\infty}^{\infty} e^{ixt} \mathbf{J}_x(s) dx,$$

which is no less "natural" than the pair (7), (8).

Added in proof: For similar results see [**14**].

References

1. A. B. Bhatia and K. S. Krishnan, Light-scattering in homogeneous media regarded as reflexion from appropriate thermal elastic waves, Proc. Roy. Soc. London, Ser. A. 192 (1948) 181–194.

2. R. P. Boas, Jr. and C. Stutz, Estimating sums with integrals, Amer. J. Physics, 39 (1971) 745–753.

3. A. Erdélyi, et al., Higher transcendental functions, McGraw-Hill, New York, etc., 1953.

4. D. Jagerman, Bounds for truncation error of the sampling expansion, SIAM J. Appl. Math., 14 (1966) 714–723.

5. L. B. W. Jolley, Summation of series, 2d ed., Dover, New York, 1961.

6. K. S. Krishnan, A simple result in quadrature, Nature, 162 (1948) 215.

7. ———, On the equivalence of certain infinite series and the corresponding integrals, J. Indian Math. Soc., (N. S.) 12, (1948) 79–88.

8. L. H. Loomis, An introduction to abstract harmonic analysis, Van Nostrand, New York, etc., 1953.

9. B. O. Peirce, A short table of integrals, 3d ed., Ginn, Boston etc., 1929.

10. H. Pollard and O. Shisha, Variations on the binomial series, this MONTHLY, 79 (1972) 495–499.

11. E. C. Titchmarsh, Introduction to the theory of Fourier integrals, Oxford, 1937.

12. G. N. Watson, A treatise on the theory of Bessel functions, 2d ed., Cambridge, 1944.

13. A. Zygmund, Trigonometric series, 2d ed., Cambridge, 1959.

14. T. J. Osler, An integral analogue of Taylor's series and its use in computing Fourier transforms, Math. Comp., 26 (1972) 449–460.

ENGLAND WAS LOST ON THE PLAYING FIELDS OF ETON: A PARABLE FOR MATHEMATICS

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I am sure that most of you have had the experience at one time or another of discovering, in an unexpected place, an old newspaper, its pages yellowed with age. You may have found that a glance at one of the old news articles jolted your mind into a moment or two of serious reflection on how far we have come since those bygone days. I was rummaging through the attic of my imagination recently when I came across a newspaper dated May 1, 1980, its pages pale with the years not yet lived. A glance at an article I found there jolted my mind into something more than a moment of reflection on where we are going. It was, it seemed to me, ...

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