The Life of $\pi$: History and Computation
A Talk for Pi Day or Other Days

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University of Newcastle
www.huffingtonpost.com/david-h-bailey/pi-day-314-14_b_4851011.html

3.14 pm, March 14, 2014
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CARMA
Why Pi? From utility to ... normality.
Recent computations and digit extraction methods.
The Life of Pi: From this extended on line presentation we shall sample

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Outline. We will cover Some of:

1. 24. Pi’s Childhood
   - Links and References
   - Babylon, Egypt and Israel
   - Archimedes Method circa 250 BCE
   - Precalculus Calculation Records
   - The Fairly Dark Ages

2. 43. Pi’s Adolescence
   - Infinite Expressions
   - Mathematical Interlude, I
   - Geometry and Arithmetic

3. 48. Adulthood of Pi
   - Machin Formulas
   - Newton and Pi
   - Calculus Calculation Records
   - Mathematical Interlude, II
   - Why Pi? Utility and Normality

4. 79. Pi in the Digital Age
   - Ramanujan-type Series
   - The ENIAC Calculator
   - Reduced Complexity Algorithms
   - Modern Calculation Records
   - A Few Trillion Digits of Pi

5. 113. Computing Individual Digits of \( \pi \)
   - BBP Digit Algorithms
   - Mathematical Interlude, III
   - Hexadecimal Digits
   - BBP Formulas Explained
   - BBP for Pi squared — in base 2 and base 3

J.M. Borwein, Life of Pi (CARMA)
Introduction: Pi is ubiquitous

- The desire to understand $\pi$, the challenge, and originally the need, to calculate ever more accurate values of $\pi$, the ratio of the circumference of a circle to its diameter, has captured mathematicians — great and less great — for eons.

- And, especially recently, $\pi$ has provided compelling examples of computational mathematics.

Pi, uniquely in mathematics, is pervasive in popular culture and the popular imagination.

In this talk I shall intersperse a largely chronological account of $\pi$’s mathematical and numerical status with examples of its ubiquity.
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The Life of Pi Teaches a Great Deal:

We shall learn that scientists are humans and see a lot:

- of important mathematics;
- of its history and philosophy;
- about the evolution of computers and computation;
- of general history, philosophy and science;
- proof and truth (certainty and likelihood);
- of just plain interesting — sometimes weird — stuff.

"Because it's not there."
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Mnemonics for Pi Abound: Piems — Word lengths give digits

Now I, even I, would celebrate
(3 1 4 1 5 9)
In rhymes inapt, the great
(2 6 5 3 5)
Immortal Syracusan, rivaled nevermore,
Who in his wondrous lore,
Passed on before
Left men for guidance
How to circles mensurate.

“When you’re young, it comes naturally, but when you get a little older, you have to rely on mnemonics.”

– punctuation is always ignored
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Yann Martel’s 2002 Booker Prize novel starts

‘‘My name is _Piscine Molitor Patel_ known to all as Pi Patel
For good measure I added \( \pi = 3.14 \)
and I then drew a large circle
which I sliced in two with a diameter, to evoke that basic lesson of geometry.’’

- **1706.** Notation of \( \pi \) introduced by William Jones.
- **1737.** Leonhard Euler (1707-83) popularized \( \pi \).
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Wife of Pi (2013)
Life of Pi (2014)

I've got to go. My mom only uses my full name when I'm in big trouble.
• Berggren, Borwein and Borwein, 3rd Ed, Springer, 2004. (3,650 years of copyright releases. *E-rights* for Ed. 4 are in process.)
  - MacTutor at www-gap.dcs.st-and.ac.uk/~history (my home town) is a good informal mathematical history source.
  - See also www.cecm.sfu.ca/~jborwein/pi_cover.html.
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Keanu Reeves, Neo, only has 314 seconds to enter “The Source.”
(Do we need Parts 4 and 5?)

From http://www.freakingnews.com/Pi-Day-Pictures--1860.asp
Pi the Movie (1998): a Sundance screenplay winner

Roger Ebert gave the film 3.5 stars out of 4: “Pi is a thriller. I am not very thrilled these days by whether the bad guys will get shot or the chase scene will end one way instead of another. You have to make a movie like that pretty skillfully before I care.

“But I am thrilled when a man risks his mind in the pursuit of a dangerous obsession.”
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Pi to one MILLION decimal places

3.14159265358979323846264338327950288419716939937510582097494459230781640628620898866280348251421170679
82148086513282306647083484609550682231725359408128481117450284102701938528210559569464228895884303196
44288109756593344652847564823738783165271230190415645866923604346104531631286428193396705620299141273
724587066650351588158488152092068229540197153636789259036001130350548820466523184414695941451116094
33052703575959195309218611738193261179310511854807446237996274955735188575272549812279381301194912
983673636324406566430860213194946395224737190721796069437020770539217176297375263846748184676693501512
0005681271425635608827857713472578986091736711782146448091224953430146549585371010507922796892592325
40219956111212901096804364481859136297747173013960518721213499999832797804995105973173281609631859
204224954531669908203624252320825334685035261931188171010003137837582865875332028381420617177669147303
5898251490287554687311195652863882353787593715578718578805321172668066130192878766111959092164201989
3808925872018654858632786859531538318279682330195203530185296899557362259941381249727157523479131151
557489724244544506958059289331166621727858548957598383754637449391392555640092770167113905848824012
8536163553670766104710181942455596198946767837494449825379774727684710410475134646220804668482590694912
933131770829891512047521625069660240580381501935125338243003558674204847647326391419272604269922297
678235478163160039413721641219924588853150208182974555670764983854594585869299590927210797509302955
3211653446972027559602364806545991398183479775356635980742654252762551814175744672890977727938000
31467060106145249219273172477325010441497356586584161161573525521334754741849468385325323290739414333
45477624168625180983569585620299122282182472550525425668767179049460165383668049886282523719786085784383
8279796766814510109538378363609560800642251522505117392989486084128488626694650424219562850222106611863
0674427862320919445047123173786969561634375191724846766465753769241389085832645998133904778027590090
94657640789512694683983325975982852262025249849077226719478268842826014767999602640136934745530568203

From 3.141592653589793238462643383279502884197169399375105820974944592.com/ This 2005 URL seems to have disappeared.
π Day turns 26: Our book Pi and the AGM is 27

- From www.google.com/trends?q=Pi+
  - H, E, D, C: “Pi Day March 14 (3.14, get it?)”
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Crossword Pi — NYT March 14, 2007

- To solve the puzzle, first note that the clue for 28 DOWN is March 14, to Mathematicians, to which the answer is PIDAY. Moreover, roughly a dozen other characters in the puzzle are $\pi = \text{PI}$.
- For example, the clue for 5 down was More pleased with the six character answer HAP$\pi$ER.

\[ \cdots - - - - - - - - - - - - - - - - \cdots \]

(MSNBC Thanksgiving 1997)
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Borweins and Plouffe  Pi Art  A Fine Book  Puzzle

(MSNBC Thanksgiving 1997)
The Puzzle (By Permission)

The New York Times Crossword
Edited by Will Shortz
No. 0314

Across
1. Enlighten
6. A couple CBS spinoffs
19. 1972 Broadway musical
14. Metal giant
15. Evict
16. Area
17. Surface again, as a road
18. Pirate or Padre, briefly
19. Camera feature
20. Barracks artwork, perhaps
22. River to the Ligurian Sea
23. Keg necessity
24. "... he drove out of sight"
25. __ St. Louis, Ill.
27. Preen
29. Greek peak

33. Vice president after Hubert
36. Patient wife of Sir Gerard
38. Action to an ante
39. Gain ___
40. French artist Odilon
42. Grape for winemaking
43. Single-dish meal
45. Broad valley
46. See 21-Down
47. Artery inserts
49. Offspring
51. Mexican mouse catcher
53. Medical procedure, in brief
54. "Wheel of Fortune" option
57. Animal with striped legs
63. It gets bigger at night
64. "Hold your horses!"
65. Idiots
66. Europe/Asia border river
67. Suffix with launder
68. Learning
69. Brownback and Obama, e.g.
70. Rick with the 1976 #1 hit "Disco Duck"
71. Yegg's targets

Down
1. Mastodon trap
2. "Mefistofele" soprano
3. Misbehave
4. Pen
5. More pleased
6. Treated with disdain
7. Enterprise crewman
8. Phone feeder
9. Many a webcast
10. Mushroom, for one
11. Unfortunate
12. Nevada's state tree
13. Disney fish
21. Colonial figure with 46-Across
26. Poker champion Ungar
27. Self-medicating excessively
28. March 14, to mathematicians
30. Book part
31. Powder, e.g.
32. 007 and others: Abbr.
33. Drains
34. Stone feature
35. Feet per second, e.g.
36. Italian range
37. Prefix with surgery
41. Captain's announcement, for short
44. Tucked away
48. Stealthy fighters
49. Sedate
50. Letter feature
54. Religious artwork
55. Jam
56. settles in
57. Symphony or sonata
58. Japanese city bombed in W.W. II
59. Bee (ge.)
61. Evening, in ads
62. Religious artwork

For answers, call 1-900-285-5656, $1.20 a minute; or, with a credit card, 1-800-814-5554.
Annual subscriptions are available for the best of Sunday crosswords from the last 50 years: 1-888-7-ACROSS.
Online subscriptions: Today's puzzle and more than 2,000 past puzzles, nytimes.com/crosswords ($34.95 a year).

J.M. Borwein
Life of Pi (CARMA)
The Puzzle Answered

ANSWER TO PREVIOUS PUZZLE

```
TEACH  CSIS   π  P π N
ALCOA OUST ZONE
RETOP NLER ZOOM
π NUP PICTyre ARNO
TAPER EAST
PRIMP MTOSSA
S π RO ENID UP π N
ALAP REDON π NOT
POT P E DALE NEWS
STENTS YOUNG
GATO MRI S π N
OKA π O π NION π ECE
PU π L WAIT JERKS
URAL ETTE ATILT
SENS DEES SAFES
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24. Pi’s Childhood
43. Pi’s Adolescence
48. Adulthood of Pi
79. Pi in the Digital Age
113. Computing Individual Digits of \( \pi \)

The Simpsons (Permission refused by Fox)

Apu: I can recite \( \pi \) to 40,000 places. The last digit is 1. Homer: Mmm... pie. (“Marge in Chains.” May 6, 1993)

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National Pi Day 3.12.2009: The first successful Pi Law

H.RES.224

Latest Title: Supporting the designation of Pi Day, and for other purposes.


Cosponsors (15)

Latest Major Action: 3/12/2009 Passed/agreed to in House. Status: On motion to suspend the rules and agree to the resolution Agreed to by the Yeas and Nays: (2/3 required): 391 - 10 (Roll no. 124).

2007-2011. Chairman of House Committee on Science and Technology.

1897. Indiana Bill 246 was fortunately shelved. Attempt to legislate value(s) of Pi and charge royalties started in the ‘Committee on Swamps’.

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**Sponsor:** Rep Gordon, Bart [TN-6] (introduced 3/9/2009)

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**Life of Pi (CARMA)**

Caption: To celebrate Pi Day 2009, the San Francisco Exploratorium made a Pi ring with more than 6,000 colored beads on it, each color representing a digit from 0 to 9.

(Credit: Daniel Terdiman/CNET)

Washington politicians took time from bailout and earmark-laden spending packages on Wednesday for what might seem like an unusual act: officially designating a National Pi Day.

That’s Pi as in ratio of a circle’s circumference-to-diameter, better known as the mathematical constant beginning with 3.14159.
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J.M. Borwein
Life of Pi (CARMA)
On Pi Day, one number 'reeks of mystery'
By Elizabeth Landau, CNN
March 12, 2010 12:36 p.m. EST

Sunday is Pi Day, on which math geeks celebrate the number representing the ratio of circumference to diameter of a circle.

(CNN) -- The sound of meditation for some people is full of deep breaths or gentle humming. For Marc Umile, it's "3.14159265358979..."

Whether in the shower, driving to work, or walking down the street, he'll mentally rattle off digits of pi to pass the time. Holding 10th place in the world for pi memorization -- he typed out 15,314 digits from memory in 2007 -- Umile meditates through one of the most beloved and mysterious numbers in all of mathematics.
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STORY HIGHLIGHTS

Pi Day falls on March 14, which is also Albert Einstein's birthday.

The true "randomness" of pi's digits -- 3.14 and so on -- has never been proven.

The U.S. House passed a resolution supporting Pi Day in March 2009.

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Judge rules "Pi is a non-copyrightable fact" on 3.14.2012

Two of many cartoons

US judge rules that you can't copyright pi

By Stephen Ornes

Video: What pi sounds like

The mathematical constant pi continues to infinity, but an extraordinary lawsuit that centered on this most beloved string of digits has come to an end. Appropriately, the decision was made on Pi Day.

On 14 March, which commemorates the constant that begins 3.14, US district court judge Michael H. Simon dismissed a claim of copyright infringement brought by one mathematical musician against another, who had also created music based on the digits of pi.

"Pi is a non-copyrightable fact, and the transcription of pi to music is a non-copyrightable idea," Simon wrote in his legal opinion dismissing the case.

"The resulting pattern of notes is an expression that meets with the non-copyrightable idea of putting pi to music."

The bizarre tale began a year ago, when Michael Blake of Portland, Oregon, released a song and YouTube video featuring an original musical composition, "What pi sounds like", translating the constant's first few dozen digits into musical notes. On Pi Day 2011, the number of page views skyrocketed as the video went viral. Now Scientist was among those who
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Google (29-1-13) and US Gov’t (14-8-12) still both love π
π Records Always Make The News

By now you get the idea: $\pi$ is everywhere ... also volumes, areas, lengths, probabilities, everywhere.

J.M. Borwein

Life of Pi (CARMA)
25. Links and References

1. The Pi Digit site: http://carma.newcastle.edu.au/bbp
2. Dave Bailey’s Pi Resources: http://crd.lbl.gov/~dhbailey/pi/

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The Infancy of Pi: Babylon, Egypt and Israel

2000 BCE. Babylonians used the approximation $\frac{22}{7} = 3.125$.

1650 BCE. Rhind papyrus: a circle of diameter nine has the area of a square of side eight:

$$\pi = \frac{256}{81} = 3.1604 \ldots$$

- *Pi* is the only topic from the earliest strata of mathematics being actively researched today.

Some argue ancient Hebrews used $\pi = 3$:

Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. (I Kings 7:23; 2 Chron. 4:2)

- More interesting is that Moses ben Maimon Maimonedes (the ‘Rambam’) (1135-1204) writes in *The true perplexity* that because of its nature “nor will it ever be possible to express it [π] exactly.”
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There are two Pi(es): Did they tell you?

Archimedes of Syracuse (c.287 – 212 BCE) was first to show that the “two Pi’s” are one in *Measurement of the Circle* (c.250 BCE):

\[
\text{Area} = \pi_1 r^2 \quad \text{and} \quad \text{Perimeter} = 2 \pi_2 r.
\]

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let \(ABCD\) be the given circle, \(K\) the triangle described.

\[\pi = 3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679821480865132823066470938446095505822317253594081284811174502841027019385211055596\]

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Archimedes Method circa 250 BCE

The first rigorous mathematical calculation of $\pi$ was also due to Archimedes, who used a brilliant scheme based on **doubling inscribed and circumscribed polygons**

$$6 \rightarrow 12 \rightarrow 24 \rightarrow 48 \rightarrow 96$$

to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

- Archimedes’ scheme is the *first true algorithm for* $\pi$, in that it is capable of producing an arbitrarily accurate value for $\pi$.  

![Graphical representation of Archimedes' method](image-url)
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Where Greece Was: Magna Graecia

- Syracuse
- Troy
- Byzantium
- Constantinople
- Rhodes (Helios)
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- Ephesus (Artemis)
- Athens (Zeus)

The others of the Seven Wonders: Lighthouse of Alexandria, Pyramids of Giza, Gardens of Babylon
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- **1906.** Discovery of a 10th-C palimpsest in Constantinople.
  - Sometime before April 14 **1229**, partially erased, cut up, and overwritten by religious text.
  - After **1929.** Painted over with gold icons and left in a wet bucket in a garden.
  - **1998.** Bought at auction for $2 million.
  - **1998-2008.** “Reconstructed” using very high-end mathematical imaging techniques.
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“Archimedes used knowledge of levers and centres of gravity to envision ways of balancing geometric figures against one another which allowed him to compare their areas or volumes. He then used rigorous geometric argument to prove *Method* discoveries.”

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Archimedes from *The Method*

“... certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. **But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.”
Let’s be Clear: $\pi$ Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* ‘knows’ this since

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1 + x^2} \, dx = \frac{22}{7} - \pi,$$  \hspace{1cm} (1)

though it would be prudent to ask ‘why’ it can perform the integral and ‘whether’ to trust it?

**Assume we trust it.** Then the integrand is strictly positive on $(0, 1)$, and the answer in (1) is an area and so strictly positive, despite millennia of claims that $\pi$ is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to $\pi$. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$
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Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

\[
\begin{align*}
    a_{n+1} &= \frac{2a_nb_n}{a_n + b_n} \quad (H) \\
    b_{n+1} &= \sqrt{a_{n+1}b_n} \quad (G)
\end{align*}
\]

These tend to $\pi$, error decreasing by a factor of four at each step.

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Proving $\pi$ is not $\frac{22}{7}$

In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_{0}^{t} \frac{x^4 (1 - x)^4}{1 + x^2} \, dx = \frac{1}{7} t^7 - \frac{2}{3} t^6 + t^5 - \frac{4}{3} t^3 + 4 t - 4 \arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus proves (1). QED

One can take this idea a bit further. Note that

$$\int_{0}^{1} x^4 (1 - x)^4 \, dx = \frac{1}{630}. \quad (2)$$
Proving $\pi$ is not $\frac{22}{7}$

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$$\int_0^1 x^4 (1-x)^4 \, dx = \frac{1}{630}. \quad (2)$$
Hence

\[ \frac{1}{2} \int_0^1 x^4 (1 - x)^4 \, dx < \int_0^1 \frac{(1 - x)^4 x^4}{1 + x^2} \, dx < \int_0^1 x^4 (1 - x)^4 \, dx. \]

Combine this with (1) and (2) to derive:

\[ 223/71 < \pi < 22/7 - 1/630 < \pi < 22/7 - 1/1260 < 22/7 \]

and so re-obtain Archimedes’ famous

\[ 3 \frac{10}{71} < \pi < 3 \frac{10}{70}. \]
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2\pi = 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},
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Kuhnian ‘Paradigm Shifts’ and Normal Science

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### Precalculus $\pi$ Calculations

<table>
<thead>
<tr>
<th>Name</th>
<th>Year</th>
<th>Digits</th>
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</thead>
<tbody>
<tr>
<td>Babylonians</td>
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<td>1</td>
</tr>
<tr>
<td>Egyptians</td>
<td>2000? BCE</td>
<td>1</td>
</tr>
<tr>
<td>Hebrews (1 Kings 7:23)</td>
<td>550? BCE</td>
<td>1</td>
</tr>
<tr>
<td><strong>Archimedes</strong></td>
<td>250? BCE</td>
<td>3</td>
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<td>Ptolemy</td>
<td>150</td>
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<tr>
<td>Liu Hui</td>
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</tr>
<tr>
<td>Tsu Ch’ung Chi</td>
<td>480?</td>
<td>7</td>
</tr>
<tr>
<td>Al-Kashi</td>
<td>1429</td>
<td>14</td>
</tr>
<tr>
<td>Romanus</td>
<td>1593</td>
<td>15</td>
</tr>
<tr>
<td>Van Ceulen (Ludolph’s number*)</td>
<td>1615</td>
<td>35</td>
</tr>
</tbody>
</table>

* Used $2^{62}$-gons for 39 places/35 correct — published posthumously.
Ludolph’s Rebuilt Tombstone in Leiden

Ludolph van Ceulen (1540-1610)

- Destroyed several centuries ago; the plans remained.
Ludolph’s Reconstructed Tombstone in Leiden

  - Attended by Dutch royal family and 750 others.
  - My brother lectured on Pi from halfway up to the pulpit.
Ludolph’s **Reconsecrated Tombstone in Leiden**

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Links and References
- Babylon, Egypt and Israel
- Archimedes Method circa 250 BCE
- Precalculus Calculation Records
- The Fairly Dark Ages
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Arithmetic was Hard

- The prior difficulty of arithmetic\(^2\) is shown by ‘college placement’ advice to a wealthy 16C German merchant:

  *If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division — assuming that he has sufficient gifts — then you will have to send him to Italy.*
  — George Ifrah or Tobias Danzig

\(^2\)**Claude Shannon** (1913-2006) had ‘Throback 1’ built to compute in Roman, at Bell Labs in 1953.
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Google Buys \((\text{Pi-3}) \times 100,000,000\) Shares

August 19, 2005

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By JOHN MARKOFF

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Google, which raised $1.67 billion in its initial public offering last August, expects to collect $4.04 billion by selling 14,159,265 million Class A shares, based on Wednesday's closing price of $285.10. In Google's whimsical fashion, the number of shares offered is the same as the first eight digits after the decimal point in pi, the ratio of the circumference of a circle to its diameter, which starts with 3.14159265.

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**Why did Google want precisely this many pieces of the Pie?**
44. Pi’s (troubled) Adolescence

1579. Modern mathematics dawns in Viéte’s product

\[
\frac{\sqrt{2}}{2} \frac{\sqrt{2} + \sqrt{2}}{2}\frac{\sqrt{2} + \sqrt{2} + \sqrt{2}}{2} \ldots = \frac{2}{\pi}
\]  

— considered to be the first truly infinite formula — and in the first continued fraction given by Lord Brouncker (1620-1684):

\[
\frac{2}{\pi} = \frac{1}{1 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \ldots}}}}
\]

---

J.M. Borwein  Life of Pi (CARMA)
Wallis Product

Eqn. (4) was based on John Wallis’ (1613-1706) ‘interpolated’ product:

\[
\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdots = \prod_{k=1}^{\infty} \frac{4k^2 - 1}{4k^2} = \frac{2}{\pi}
\]

which led to discovery of the Gamma function and much more.

- Christiaan Huygens (1629-1695) did not believe (5) before he checked it numerically.

It’s a clue.
A never repeating or ending chain, the total timeless catalogue, elusive sequences, sum of the universe.
This riddle of nature begs:
Can the totality see no pattern, revealing order as reality’s disguise?
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Mathematical Interlude I: the Zeta Function

Formula (5) follows from Euler’s product formula for $\pi$,

$$\frac{\sin(\pi x)}{x} = c \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2} \right)$$

(6)

with $x = 1/2$, or by integrating $\int_0^{\pi/2} \sin^{2n}(t) \, dt$ by parts.

One may divine (6) — as Euler did — by considering $\sin(\pi x)$ as an ‘infinite’ polynomial and obtaining a product in terms of the roots $0, \{1/n^2\}$. Euler argued that, like a polynomial, $c = \pi$ is the value at 0. The coefficient of $x^2$ in the Taylor series is the sum of the roots:

$$\zeta(2) := \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hence, $\zeta(2n)$ is rational $\times \pi^{2n}$: so

$$\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$$

(using Bernoulli numbers)

1976. Apéry showed $\zeta(3)$ irrational; and Zudilin (CARMA) has shown at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.
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CARMA
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Final Jeopardy! 20 Sept 2005: Mnemonics are valuable

**CATEGORY:** By the numbers.  **CLUE:** The phrase “How I want a drink, alcoholic of course” is often used to help memorize this.

**ANSWER:** What is Pi?  **FINAL SCORES:**

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2.14-2.16.2011 IBM *Watson* query system (now an oncologist) routed Jeopardy champs Jennings & Rutter:  
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I am ashamed to tell you to how many figures I carried these computations, having no other business at the time. Isaac Newton, 1666

- 17C Newton and Leibnitz discovered calculus ... and fought over priority (Machin adjudicated).
- It was instantly exploited to find formulas for $\pi$.

One early use comes from the $\arctan$ integral and series:\(^3\)

$$
\tan^{-1} x = \int_0^x \frac{dt}{1 + t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots) \, dt
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= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots
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\(^3\)Known to Madhava of Sangamagrama (c. 1350 – c. 1425) near Kerala. He probably computed 13 digits of Pi.
I am ashamed to tell you to how many figures I carried these computations, having no other business at the time. Isaac Newton, 1666

- **17C** Newton and Leibnitz discovered calculus ... and fought over priority (Machin adjudicated).
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Madhava–Gregory–Leibniz formula

Formally \( x := 1 \) gives the Gregory–Leibniz formula (1671–74)

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\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots
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- Naively, this is useless — hundreds of terms produce two digits.
- Sharp guided by Edmund Halley (1656-1742) used \( \tan^{-1}(1/\sqrt{3}) \)
• By contrast, Euler’s (1738) trigonometric identity

\[
\tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)
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produces the geometrically convergent:

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John Machin (1680-1751) and Brook Taylor (1685-1731)

An even faster formula, found earlier by John Machin — Brook Taylor’s teacher — lies in the identity

\[
\frac{\pi}{4} = 4 \tan^{-1} \left( \frac{1}{5} \right) - \tan^{-1} \left( \frac{1}{239} \right). \tag{9}
\]

- Used in numerous computations of \(\pi\) (starting in 1706) culminating with Shanks’ computation of \(\pi\) to 707 decimals in 1873.
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Newton discovered a different (disguised arcsin) formula. He considered the area $A$ of the red region to the right:

Now $A := \int_{0}^{1/4} \sqrt{x - x^2} \, dx$ equals the circular sector, $\pi/24$, less the triangle, $\sqrt{3}/32$. His new binomial theorem gave:

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- As noted, he ‘apologized’ for “having no other business at the time.” A standard 1951 MAA chronology said, condescendingly, “*Newton never tried to compute* $\pi$.”

Newton, Gregory (1638-1675) and Leibniz (1646-1716)

The fire of London ended the plague in September 1666. The plague closed Cambridge and left Newton free at his country home to think.

*Wikipedia:* Newton made revolutionary inventions and discoveries in calculus, motion, optics and gravitation. As such, it has later been called Isaac Newton’s “Annus Mirabilis.”
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Calculus $\pi$ Calculations: and an IBM 7090

<table>
<thead>
<tr>
<th>Name</th>
<th>Year</th>
<th>Digits</th>
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<tr>
<td>Sharp (and Halley)</td>
<td>1699</td>
<td>71</td>
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<td>Machin</td>
<td>1706</td>
<td>100</td>
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<td>Strassnitzky and Dase</td>
<td>1844</td>
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<td>Rutherford</td>
<td>1853</td>
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<td>W. Shanks</td>
<td>1874</td>
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<td>Ferguson (Calculator)</td>
<td>1947</td>
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<td>1949</td>
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<td>Genuys</td>
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<td>Guilloud and Bouyer</td>
<td>1973</td>
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</table>
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An early vegetarian (who *misused needles*) next to the inventor of **Monte Carlo** methods.

1. Draw a unit square and inscribe a circle within: the area of the circle is \( \frac{\pi}{4} \).

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Monte Carlo Methods

- This is a Monte Carlo estimate (MC) for $\pi$.
- MC simulation: slow ($\sqrt{n}$) convergence — but great in parallel on Beowulf clusters.
- Used in Manhattan project ... the atomic-bomb predates digital computers!
Gauss (1777-1855), Johan Dase and William Shanks

In his teens, Viennese computer and 'kopfrechner' Dase (1824-1861) publicly demonstrated his skill by multiplying

\[79532853 \times 93758479 = 7456879327810587\]

- in 54 seconds; 20-digits in 6 min; 40-digits in 40 min; 100-digit numbers in \(8\frac{3}{4}\) hours etc.
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In 1849-50 Dase made a seven-digit Tafel der natürlichen Logarithmen der Zahlen, asking the Hamburg Academy to fund factorization of integers between 7 and 10 million (evidence for the Prime Number Theorem).

- Now Gauss was impressed and recommended Dase be funded.
- 1861. When Dase died he had only reached 8,000,000.

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The Three **Construction Problems of Antiquity**

The other two are doubling the cube and trisecting the angle.

- This settled *once and for all*, the ancient Greek question of whether the circle could be squared with ruler and compass.
- It cannot, because lengths of lines that can be constructed using ruler and compasses (*constructible numbers*) are necessarily algebraic, and squaring the circle is equivalent to constructing the value of $\pi$.
- **Aristophanes** (448-380 BCE) ‘knew’ this and derided all ‘circle-squarers’ in his play *The Birds* of 414 BCE.

τετραγωσιειν

*Life of Pi (CARMA)*
Ivan Niven’s 1947 proof that $\pi$ is irrational. Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since $n! f(x)$ has integral coefficients and terms in $x$ of degree not less than $n$, $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x = 0$; also for $x = \pi = a/b$, since $f(x) = f(a/b - x)$. By elementary calculus we have

$$\frac{d}{dx} \{ F'(x) \sin x - F(x) \cos x \}$$

$$= F''(x) \sin x + F(x) \sin x = f(x) \sin x$$
The Irrationality of $\pi$, II

and

$$\int_0^\pi f(x) \sin x \, dx = \left[ F'(x) \sin x - F(x) \cos x \right]_0^\pi$$

$$= F(\pi) + F(0). \quad (10)$$

Now $F(\pi) + F(0)$ is an integer, since $f^{(j)}(0)$ and $f^{(j)}(\pi)$ are integers. But for $0 < x < \pi$,

$$0 < f(x) \sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (10) is positive but arbitrarily small for $n$ sufficiently large. Thus (10) is false, and so is our assumption that $\pi$ is rational.

QED

- This, exact transcription of Niven’s proof, is an excellent intimation of more sophisticated irrationality and transcendence proofs.
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• At the end of his story, **Piscine (Pi) Molitor** writes

I am a person who believes in form, in harmony of order. Where we can, we must give things a meaningful shape. For example — I wonder — could you tell my jumbled story in exactly one hundred chapters, not one more, not one less? I’ll tell you, that’s one thing I hate about my nickname, the way that number runs on forever. It’s important in life to conclude things properly. Only then can you let go.

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Summation. Why Pi? “Pi is Mount Everest.”

What motivates modern computations of $\pi$ — given that irrationality and transcendence of $\pi$ were settled a century ago?

- One motivation is the raw challenge of harnessing the stupendous power of modern computer systems.

Programming is quite hard — especially on large, distributed memory computer systems: load balancing, communication needs, etc.

Substantial practical spin-offs accrue:

- Accelerating computations of $\pi$ sped up the fast Fourier transform (FFT) — heavily used in science and engineering.
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- Beyond practical considerations are fundamental issues such as the normality (digit randomness and distribution) of $\pi$.

  John von Neumann so prompted ENIAC computation of $\pi$ and $e$ — and $e$ showed anomalies.

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Pi Seems Normal: Things we sort of know about Pi

A walk on a billion hex digits of Pi with *box dimension* 1.85343...

- A 100Gb 100 billion step walk is at [http://carma.newcastle.edu.au/walks/](http://carma.newcastle.edu.au/walks/)
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Pi Seems Normal: Some million bit comparisons

Euler’s constant and a pseudo-random number

Fractal dimension of 10,000 random walks of 1 million steps

Fractal dimension of random walks of 1 million steps

To 1 million bits, $\pi$ has dim 1.7542 and $\gamma$ has dim 1.77343

J.M. Borwein
Life of Pi (CARMA)
Pi Seems Normal: Comparisons to Stoneham’s number $\sum_{k \geq 1} 1/(3^k \cdot 2^k)$, I

In base 2 Stoneham’s number is provably normal. It may be normal base 3.
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Stoneham’s number is provably abnormal base 6 (too many zeros).
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Pi Seems Normal: Comparisons to Human Genomes — we are base 4 no’s

Chromosome X

\[ c = [1, 0] \]
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Chromosome 1

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The X Chromosome (34K) and Chromosome One (10K).
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Pi Seems Normal: Comparisons to other provably normal numbers

Erdős-Copeland number (base 2) and Champernowne number (base 10).

All pictures are thanks to Fran Aragon and Jake Fountain
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Pi is Still Mysterious: Things we don’t know about Pi

We do not ‘know’ (in the sense of being able to prove) whether ....

- The simple continued fraction for Pi is unbounded.
  - Euler found the 292.
- There are infinitely many sevens in the decimal expansion of Pi.
- There are infinitely many ones in the ternary expansion of Pi.
- There are equally many zeroes and ones in the binary expansion of Pi.
- Or pretty much anything I have not told you.

\[
\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}\
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### Decimal Digit Frequency: and “Johnny” von Neumann

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**JvN (1903-57) at the Institute for Advanced Study**

---

1st von Neumann architecture machine
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Hexadecimal Digit Frequency: and Richard Crandall (Apple HPC)

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(1947–2012)
Changing Cognitive Tastes

Why in antiquity $\pi$ was not measured to greater accuracy than $22/7$ (with rope)?

It reflects not an inability, but rather a very different mindset to a modern (Baconian) experimental one — see Francis Bacon’s *De augmentis scientiarum* (1623).

- Gauss and Ramanujan did not exploit their identities for $\pi$.
- An algorithm, as opposed to a closed form, was unsatisfactory to them — especially Ramanujan. He preferred

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\frac{3}{\sqrt{163}} \log (640320) \approx \pi \quad \text{and} \quad \frac{3}{\sqrt{67}} \log (5280) \approx \pi
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Gourevich used integer relation computer methods to find the Ramanujan-type series — discussed below — in (11):

\[
\frac{4}{\pi^3} = \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n+1}
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(11)

where \( r(n) := \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} \).

• I can “discover” it using 30-digit arithmetic and check it to 1,000 digits in 0.75 sec, 10,000 digits in 4.01 min with two naive command-line instructions in Maple.
  - No one has any inkling of how to prove it.
  - I “know” the beautiful identity is true — it would be more remarkable were it eventually to fail.
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Changing Cognitive Tastes: **Truth without Proof**

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Pi in High Culture (1993)

The admirable number pi:
three point one four one.
All the following digits are also initial,
five nine two because it never ends.
It can’t be comprehended six five three five at a glance,
eight nine by calculation,
seven nine or imagination,
not even three two three eight by wit, that is, by comparison
four six to anything else
two six four three in the world.
The longest snake on earth calls it quits at about forty feet.
Likewise, snakes of myth and legend, though they may hold out a bit longer.
The pageant of digits comprising the number pi doesn’t stop at the page’s edge.
It goes on across the table, through the air, over a wall, a leaf, a bird’s nest, clouds, straight into the sky, through all the bottomless, bloated heavens.

1996 Nobel Wislawa Szymborska (2-7-1923  1-2-2012)

Oh how brief - a mouse tail, a pigtail - is the tail of a comet!
How feeble the star’s ray, bent by bumping up against space!
While here we have two three fifteen three hundred nineteen
my phone number your shirt size the year nineteen hundred and seventy-three the sixth floor the number of inhabitants sixty-five cents your hip measurement two fingers a charade, a code, in which we find hail to thee, blithe spirit, bird thou never wert alongside ladies and gentlemen, no cause for alarm, as well as heaven and earth shall pass away, but not the number pi, oh no, nothing doing, it keeps right on with its rather remarkable five, its uncommonly fine eight, its far from final seven, nudging, always nudging a sluggish eternity to continue.
24. Pi’s Childhood
43. Pi’s Adolescence
48. Adulthood of Pi
79. Pi in the Digital Age
113. Computing Individual Digits of $\pi$

Pi in **High Culture (1993)**

The admirable number pi:

*three point one four one.*

All the following digits are also initial,

*five nine two* because it never ends.

It can’t be comprehended *six five three five* at a glance,

*eight nine* by calculation,

*seven nine* or imagination,

not even *three two three eight* by wit, that is, by comparison

*four six* to anything else

*two six four three* in the world.

The longest snake on earth calls it quits at about forty feet.

Likewise, snakes of myth and legend, though they may hold out a bit longer.

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*my phone number your shirt size the year nineteen hundred and seventy-three the sixth floor the number of inhabitants sixty-five cents hip measurement two fingers a charade, a code, in which we find hail to thee, blithe spirit, bird thou never wert*

alongside *ladies and gentlemen, no cause for alarm, as well as heaven and earth shall pass away, but not the number pi, oh no, nothing doing, it keeps right on with its rather remarkable five, its uncommonly fine eight, its far from final seven, nudging, always nudging a sluggish eternity to continue.*

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Computers Cease Being Human

1950s. Commercial computers — and discovery of advanced algorithms for arithmetic — unleashed \( \pi \).

1965. The new fast Fourier transform (FFT) performed high-precision multiplications much faster than conventional methods — viewing numbers as polynomials in \( \frac{1}{10} \).

- Newton methods helped reduce time for computing \( \pi \) to ultra-precision from millennia to weeks or days.

\[
x \leftarrow x + x(1 - bx)
\]

converts \( \frac{1}{b} \) to \( 4 \times \)

\[
x \leftarrow x + x(1 - ax^2)/2
\]

converts \( \frac{1}{\sqrt{a}} \) to \( 6 \times \) (7 for \( \sqrt{a} \))

▽ But until the 1980s all computer evaluations of \( \pi \) employed classical formulas, usually of Machin-type.

Happily, MRI and FFT were discovered at the same time.
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\[
\begin{align*}
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\text{converts } 1/b & \text{ to } 4 \times
\end{align*}
\]

\[
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Newton’s method is self-correcting and quadratically convergent.

So we start close (to the left); and

We keep only the first half of each answer.
Newton Method Illustrated in Maple for $1/7$

> restart: Digits := 100; N := x -> x + x*(1 - 7*x);

\[ N := x \rightarrow x + x(1 - 7x) \]

> Digits := 64; x := 0.142; for k from 1 to 6 do x := evalf(N(x), 2^(k)+2); od;

\[
\begin{align*}
  x &:= 0.142 \\
  x &:= 0.1429 \\
  x &:= 0.142857 \\
  x &:= 0.1428571429 \\
  x &:= 0.1428571428571428571428571428571429
\end{align*}
\]

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Ramanujan’s Seventy-Fifth Birthday Stamp.

- Truly new infinite series formulas were discovered by the self-taught Indian genius Srinivasa Ramanujan around 1910.
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Ramanujan Series for $\frac{1}{\pi}$  

One of these series is the remarkable:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}$$  \hspace{1cm} (12)

- Each term adds an additional eight correct digits.

1985. ‘Hacker’ Bill Gosper used (12) to compute 17 million digits of (the continued fraction for) $\pi$; and so the first proof of (12)!

1987. David and Gregory Chudnovsky found a variant:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}$$  \hspace{1cm} (13)

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Some Series Can Save Significant Work

- Relatedly, the Ramanujan-type series:

\[
\frac{1}{\pi} = \sum_{n=0}^{\infty} \left( \frac{\binom{2n}{n}}{16^n} \right)^3 \frac{42n + 5}{16}. \tag{14}
\]

allows one to compute the billionth binary digit of \(1/\pi\), or the like, \textit{without computing the first half} of the series.

Conjecture (Moore’s Law in Electronics Magazine 19 April, 1965)

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**SIZE/WEIGHT:** ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons.

The ENIAC in the Smithsonian

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1949 ‘skunk-works’ computation of $\pi$ — suggested by von Neumann — to 2,037 places in 70 hrs.

Origin of the term ‘bug’?

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- Eckert-Mauchly Computer Corp. bought by Remington Rand which became Sperry Rand (Unisys).
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Ballantine’s (1939) Series for $\pi$

Another formula of Euler for $\arccot$ is:

$$x \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! (x^2 + 1)^{n+1}} = \arctan \left( \frac{1}{x} \right).$$

As $10 (18^2 + 1) = 57^2 + 1 = 3250$ we may rewrite the formula

$$\frac{\pi}{4} = \arctan \left( \frac{1}{18} \right) + 8 \arctan \left( \frac{1}{57} \right) - 5 \arctan \left( \frac{1}{239} \right)$$

used by Shanks and Wrench in 1961 for 100,000 digits, and by Guilloud and Boyer in 1973 for a million digits of Pi in the efficient form

$$\pi = 864 \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! 325^{n+1}} + 1824 \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! 3250^{n+1}} - 20 \arctan \left( \frac{1}{239} \right)$$

where terms of the second series are just decimal shifts of the first.
Another formula of Euler for \( \arccot \) is:

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where terms of the second series are just decimal shifts of the first.
Calculation of \( \pi \) to 100,000 Decimals

By Daniel Shanks and John W. Wrench, Jr.

1. Introduction. The following comparison of the previous calculations of \( \pi \) performed on electronic computers shows the rapid increase in computational speeds which has taken place.

<table>
<thead>
<tr>
<th>Author</th>
<th>Machine</th>
<th>Date</th>
<th>Precision</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reitwienner</td>
<td>ENIAC</td>
<td>1949</td>
<td>2037D</td>
<td>70 hours</td>
</tr>
<tr>
<td>Nicholson &amp; Jeene</td>
<td>NORC</td>
<td>1964</td>
<td>3089D</td>
<td>13 min.</td>
</tr>
<tr>
<td>Felton</td>
<td>Pegasus</td>
<td>1968</td>
<td>10000D</td>
<td>33 hours</td>
</tr>
<tr>
<td>Genius</td>
<td>IBM 704</td>
<td>1968</td>
<td>10000D</td>
<td>100 min.</td>
</tr>
<tr>
<td>Unpublished</td>
<td>IBM 704</td>
<td>1959</td>
<td>16167D</td>
<td>4.3 hours</td>
</tr>
</tbody>
</table>

All these computations, except Felton's, used Machin's formula:

\[
\pi = 16 \tan^{-1} \frac{1}{2} - 4 \tan^{-1} \frac{\sqrt{3}}{5}
\]  

Other things being equal, that is, assuming the use of the same machine and the same program, an increase in precision by a factor \( f \) requires \( f \) times as much memory, and \( f^2 \) times as much machine time. For example, a hypothetical computation of \( \pi \) to 100,000D using Genius' program would require 167 hours on an IBM 704 system and more than 28,000 words of core memory. However, since the latter is not available, the program would require modification, and this would extend the running time. Further, since the probability of a machine error would be more than 100 times that during Genius' computation, prudence would require still other program modifications, and, therefore, still more machine time.

5. A Million Decimals? Can \( \pi \) be computed to 1,000,000 decimals with the computers of today? From the remarks in the first section we see that the program which we have described would require times of the order of months. But since the memory of a 7090 is too small, by a factor of ten, a modified program, which writes and reads partial results, would take longer still. One would really want a computer 100 times as fast, 10 times as reliable, and with a memory 10 times as large. No such machine now exists.

There are, of course, many other formulas similar to (1), (2), and (5), and other programming devices are also possible, but it seems unlikely that any such modification can lead to more than a rather small improvement.

Are there entirely different procedures? This is, of course, possible. We cite the following: compute \( 1/\pi \) and then take its reciprocal. This sounds fantastic, but, in fact, it can be faster than the use of equation (2). One can compute \( 1/\pi \) by Ramanujan's formula [8]:

\[
\frac{1}{\pi} = \frac{1}{4} \left( \frac{1123}{882} - \frac{22583}{882^2} + \frac{1}{2} + \frac{44043}{882^2} - \frac{1}{3} + \frac{44043}{882^2} - \frac{1}{5} - \frac{44043}{882^2} + \cdots \right).
\]

The first factors here are given by \((-1)^n (1123 + 21460n)^2\) \(n\) is an integer. A binary value of \( 1/\pi \) equivalent to 100,000D, can be computed on a 7090 using equation (6) in 6 hours instead of the 8 hours required for the application of equation (2). To reciprocate this value of \( 1/\pi \) would take about 1 hour. Thus, we can reduce the time required by (2) by an hour. But unfortunately we lose our overlapping check, and, in any case, this small gain is quite inadequate for the present question.

One could hope for a theoretical approach to this question of optimization—a theory of the "depth" of numbers—but no such theory now exists. One can guess that \( \varepsilon \) is not as "deep" as \( \pi \), but try to prove it!

Such a theory would, of course, take years to develop. In the meantime—say, in 5 to 7 years—such a computer as we suggested above (100 times as fast, 100 times as reliable, and with 10 times the memory) will, no doubt, become a reality. At that time a computation of \( \pi \) to 1,000,000D will not be difficult.

---

*We have computed \( 1/\pi \) by (6) to over 5000D in less than a minute.

† We have computed \( \pi \) on a 7090 to 100,265D by the obvious program. This takes 2.5 hours instead of the 8-hour run for \( \pi \) by (2).
Shanks (the 2nd) and Wrench: “A Million Decimals?” (1961)

There are, of course, many other formulas similar to (1), (2), but programming devices are also possible, but it seems unlikely that a single formula can lead to more than a rather small improvement.

Are there entirely different procedures? This is, of course, following: compute $1/\pi$ and then take its reciprocal. This procedure, in fact, it can be faster than the use of equation (2). One procedure is Ramanujan’s formula [8]:

$$\frac{1}{\pi} = \frac{1}{4} \left( \frac{1123}{882^2} - \frac{2582}{882^2} \cdot \frac{1}{2} + \frac{4043}{882^2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \right).$$

The first factors here are given by $(-1)^k (1123 + 21460k)$. This can be equivalent to 100,000D, which can be computed on a 7090 computer, instead of the 8 hours required for the application of equation (2). This value of $1/\pi$ would take about 1 hour. Thus, we can reach $1/\pi$ by (2) by an hour. But unfortunately we lose our overlapping case, this small gain is quite inadequate for the present question.

One could hope for a theoretical approach to this question, especially the theory of the “depth” of numbers—no such theory now. This is not as “deep” as $\pi$, but try to prove it!

Such a theory would, of course, take years to develop. It took about 5 to 7 years—not so much as we suggested above (10 times as reliable, and with 10 times the memory), no doubt. At that time a computation of $\pi$ to 1,000,000D will not be the same.

* We have computed $1/\pi$ by (6) to over 5000D in less than a month.

† We have computed $\pi$ on a 7090 to 100,265D by the obvious method of 64,000D in 40,000 hours instead of the 8-hour run for $\pi$ by (2).

By Daniel Shanks and John W. Wrench, Jr.

1. Introduction. The following comparison of the previous calculations of $\pi$ performed on electronic computers shows the rapid increase in computational speeds which has taken place.

<table>
<thead>
<tr>
<th>Author</th>
<th>Machine</th>
<th>Date</th>
<th>Precision</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reitwiesner</td>
<td>ENIAC</td>
<td>1949</td>
<td>2037D</td>
<td>70 hours</td>
</tr>
<tr>
<td>Felton</td>
<td>Pegasus</td>
<td>1958</td>
<td>10000D</td>
<td>33 hours</td>
</tr>
<tr>
<td>Genuys</td>
<td>IBM 704</td>
<td>1958</td>
<td>10000D</td>
<td>100 min.</td>
</tr>
<tr>
<td>Unpublished</td>
<td>IBM 704</td>
<td>1959</td>
<td>16167D</td>
<td>4.3 hours</td>
</tr>
</tbody>
</table>

All these computations, except Felton’s, used Machin’s formula:

$$\pi = 16 \tan^{-1} \frac{1}{5} - 4 \tan^{-1} \frac{1}{239}.$$

Other things being equal, that is, assuming the use of the same machine and the same program, an increase in precision by a factor $f$ requires $f$ times as much memory, and $f^2$ times as much machine time. For example, a hypothetical computation of $\pi$ to 100,000D using Genuys’ program would require 167 hours on an IBM 704 system and more than 38,000 words of core memory. However, since the latter is not available, the program would require modification, and this would extend the running time. Further, since the probability of a machine error would be more than 100 times that during Genuys’ computation, prudence would require still other program modifications, and, therefore, still more machine time.
A *random walk* on $\pi$ (courtesy David and Gregory Chudnovsky)

- See Richard Preston’s: “The Mountains of Pi”, *New Yorker*, March 2, 1992 (AAAS-Westinghouse Award for Science Journalism);
- A marvellous “Chasing the Unicorn” and 2005 NOVA program.
The First Million Digits of $\pi$

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Reduced Complexity Methods

These series are much faster than classical ones, but the number of terms needed still increases linearly with the number of digits.

Twice as many digits correct requires twice as many terms of the series.

1976. Richard Brent of ANU-CARMA and Eugene Salamin independently found a reduced complexity algorithm for $\pi$. 

- It takes $O(\log N)$ operations for $N$ digits.
- Uses arithmetic-geometric mean iteration (AGM) and other elliptic integral ideas due to Gauss and Legendre circa 1800.
  - Gauss — and others — missed connection to computing $\pi$. 

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CARMA
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Algorithm (Brent-Salamin AGM iteration)

Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. Calculate

$$a_k = \frac{a_{k-1} + b_{k-1}}{2} \quad (A) \quad b_k = \sqrt{a_{k-1}b_{k-1}} \quad (G)$$

$$c_k = a_k^2 - b_k^2, \quad s_k = s_{k-1} - 2^k c_k$$

and compute

$$p_k = \frac{2a_k^2}{s_k}.$$  \hspace{1cm} (15)

Then $p_k$ converges quadratically to $\pi$.

- Each step doubles the correct digits — successive steps produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 digits of $\pi$.
- 25 steps compute $\pi$ to 45 million digits. But, steps must be performed to the desired precision.
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Four Famous Pi Guys: Salamin, Kanada, Bailey and Gosper in 1987

- To appear in Donald Knuth’s book of mathematics pictures.
And Some Others: Niven, Shanks (1917-96), Brent, Zudilin (_CYCLE_SYMBOL)
1985. Peter and I discovered algebraic algorithms of all orders:

**Algorithm (Cubic Algorithm)**

Set $a_0 = 1/3$ and $s_0 = (\sqrt{3} - 1)/2$. Iterate

$$r_{k+1} = \frac{3}{1 + 2(1 - s_k^3)^{1/3}}, \quad s_{k+1} = \frac{r_k + 1}{2}$$

and

$$a_{k+1} = r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1).$$

Then $1/a_k$ converges cubically to $\pi$.

- The number of digits correct more than triples with each step.
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A Fourth Order Algorithm

Algorithm (Quartic Algorithm)

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

and

$$a_{k+1} = a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2).$$

Then $1/a_k$ converges quartically to $\pi$.

- Using $4 \times \text{‘plus’ 1 ÷ ‘plus’ 2} \div \sqrt{\cdot} = 19$ full precision $\times$ per step. So 20 steps costs out at around 400 full precision multiplications.

(This assumes intermediate storage. Additions are cheap.)
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### Modern Calculation Records: and IBM Blue Gene/L at Argonne

<table>
<thead>
<tr>
<th>Name</th>
<th>Year</th>
<th>Correct Digits</th>
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</thead>
<tbody>
<tr>
<td>Miyoshi and Kanada</td>
<td>1981</td>
<td>2,000,036</td>
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<tr>
<td>Kanada-Yoshino-Tamura</td>
<td>1982</td>
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</tr>
<tr>
<td>Gosper</td>
<td>1985</td>
<td>17,526,200</td>
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<tr>
<td>Bailey</td>
<td>Jan. 1986</td>
<td>29,360,111</td>
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<tr>
<td>Kanada and Tamura</td>
<td>Sep. 1986</td>
<td>33,554,414</td>
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<td>Kanada et. al</td>
<td>Jan. 1987</td>
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</tr>
<tr>
<td>Kanada and Tamura</td>
<td>Jan. 1988</td>
<td>201,326,551</td>
</tr>
<tr>
<td>Chudnovskys</td>
<td>May 1989</td>
<td>480,000,000</td>
</tr>
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<td>Kanada and Tamura</td>
<td>Jul. 1989</td>
<td>536,870,898</td>
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<tr>
<td>Kanada and Tamura</td>
<td>Nov. 1989</td>
<td>1,073,741,799</td>
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<tr>
<td>Chudnovskys</td>
<td>Aug. 1991</td>
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<td>Chudnovskys</td>
<td>May 1994</td>
<td>4,044,000,000</td>
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<td>Kanada and Takahashi</td>
<td>Oct. 1995</td>
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<td>Jul. 1997</td>
<td>51,539,600,000</td>
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<td>Kanada and Takahashi</td>
<td>Sep. 1999</td>
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<td>Kanada-Ushiro-Kuroda</td>
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<td>Apr. 2009</td>
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<td>Bellard</td>
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<td>Aug. 2010</td>
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<td>Kondo and Yee</td>
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<td>Kondo and Yee</td>
<td>Dec. 2013</td>
<td>12,200,000,000,000,000</td>
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Moore’s Law Marches On

Computation of $\pi$ since 1975 plotted vs. Moore’s law predicted increase.

J.M. Borwein  Life of Pi (CARMA)
An Amazing **Algebraic Approximation** to $\pi$

The **transcendental number** $\pi$ and the **algebraic number** $1/a_{20}$ actually agree for more than **1.5 trillion decimal places**.

- $\pi$ and $1/a_{21}$ agree for more than **six trillion decimal places**.

1984. I found these on a **16K** upgrade of an 8K double-precision TRS80-100 Radio Shack portable.

1986. A **29 million** digit calculation at NASA Ames — just after the shuttle disaster — uncovered **CRAY** hardware and software faults.

- Took 6 months to convince Seymour Cray; then ran on every **CRAY** before it left the factory.
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$a_0 = 6 - 4 \sqrt{2}$

$y_1 = \frac{1 - \frac{\sqrt{1 - y_0^4}}{1 + \frac{\sqrt{1 - y_0^4}}{4}}}{1 + \frac{\sqrt{1 - y_0^4}}{4}}, a_1 = a_0 \left(1 + y_1\right)^4 - 2^3 y_1 \left(1 + y_1 + y_1^2\right)$

$y_2 = \frac{1 - \frac{\sqrt{1 - y_1^4}}{1 + \frac{\sqrt{1 - y_1^4}}{4}}}{1 + \frac{\sqrt{1 - y_1^4}}{4}}, a_2 = a_1 \left(1 + y_2\right)^4 - 2^5 y_2 \left(1 + y_2 + y_2^2\right)$

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$y_5 = \frac{1 - \frac{\sqrt{1 - y_4^4}}{1 + \frac{\sqrt{1 - y_4^4}}{4}}}{1 + \frac{\sqrt{1 - y_4^4}}{4}}, a_5 = a_4 \left(1 + y_5\right)^4 - 2^{11} y_5 \left(1 + y_5 + y_5^2\right)$

$y_6 = \frac{1 - \frac{\sqrt{1 - y_5^4}}{1 + \frac{\sqrt{1 - y_5^4}}{4}}}{1 + \frac{\sqrt{1 - y_5^4}}{4}}, a_6 = a_5 \left(1 + y_6\right)^4 - 2^{13} y_6 \left(1 + y_6 + y_6^2\right)$

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$y_{10} = \frac{1 - \frac{\sqrt{1 - y_9^4}}{1 + \frac{\sqrt{1 - y_9^4}}{4}}}{1 + \frac{\sqrt{1 - y_9^4}}{4}}, a_{10} = a_9 \left(1 + y_{10}\right)^4 - 2^{21} y_{10} \left(1 + y_{10} + y_{10}^2\right)$
\[ a_0 = 6 - 4 \sqrt{2} \]

\[ y_1 = \frac{1 - 4\sqrt{1 - y_0^4}}{1 + 4\sqrt{1 - y_0^4}}, \quad a_1 = a_0 (1 + y_1)^4 - 2^3 y_1 \left( 1 + y_1 + y_1^2 \right) \]

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\[ y_4 = \frac{1 - 4\sqrt{1 - y_3^4}}{1 + 4\sqrt{1 - y_3^4}}, \quad a_4 = a_3 (1 + y_4)^4 - 2^9 y_4 \left( 1 + y_4 + y_4^2 \right) \]

\[ y_5 = \frac{1 - 4\sqrt{1 - y_4^4}}{1 + 4\sqrt{1 - y_4^4}}, \quad a_5 = a_4 (1 + y_5)^4 - 2^{11} y_5 \left( 1 + y_5 + y_5^2 \right) \]

\[ y_6 = \frac{1 - 4\sqrt{1 - y_5^4}}{1 + 4\sqrt{1 - y_5^4}}, \quad a_6 = a_5 (1 + y_6)^4 - 2^{13} y_6 \left( 1 + y_6 + y_6^2 \right) \]

\[ y_7 = \frac{1 - 4\sqrt{1 - y_6^4}}{1 + 4\sqrt{1 - y_6^4}}, \quad a_7 = a_6 (1 + y_7)^4 - 2^{15} y_7 \left( 1 + y_7 + y_7^2 \right) \]

\[ y_8 = \frac{1 - 4\sqrt{1 - y_7^4}}{1 + 4\sqrt{1 - y_7^4}}, \quad a_8 = a_7 (1 + y_8)^4 - 2^{17} y_8 \left( 1 + y_8 + y_8^2 \right) \]

\[ y_9 = \frac{1 - 4\sqrt{1 - y_8^4}}{1 + 4\sqrt{1 - y_8^4}}, \quad a_9 = a_8 (1 + y_9)^4 - 2^{19} y_9 \left( 1 + y_9 + y_9^2 \right) \]

\[ y_{10} = \frac{1 - 4\sqrt{1 - y_9^4}}{1 + 4\sqrt{1 - y_9^4}}, \quad a_{10} = a_9 (1 + y_{10})^4 - 2^{21} y_{10} \left( 1 + y_{10} + y_{10}^2 \right) \]
\[
y_{11} = \frac{1 - \sqrt[4]{1 - y_{10}^4}}{1 + \sqrt[4]{1 - y_{10}^4}}, \quad a_{11} = a_{10} (1 + y_{11})^4 - 2^{23} y_{11} \left(1 + y_{11} + y_{11}^2\right)
\]
\[
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\[
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\]
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\[
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“A Billion Digits is Impossible”

- Since 1988 used, with Salamin-Brent, by Kanada’s Tokyo team. Including: $\pi$ to 200 billion decimal digits in 1999 ... and records in 2009.

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- In 1997 the first occurrence of the sequence 0123456789 was found (late) in the decimal expansion of $\pi$ starting at the 17,387,594,880-th digit after the decimal point.

- In consequence the status of several famous intuitionistic examples due to Brouwer and Heyting has changed.
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Billions and Billions

I spent my entire fortune to buy this supercomputer.

What does it do?

It can calculate the value of pi to about a jillion decimal places...

A lot of people talk about the areas of circles, but I'm doing something about it.
Star Trek

Kirk asks:

“Are there some mathematical problems that simply can’t be solved?”

And Spock ‘fries the brains’ of a rogue computer by telling it:

“Compute to the last digit the value of ... Pi.”
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Pi the Song: from the album Aerial

2005 Influential Singer-songwriter Kate Bush sings “Pi” on Aerial.

Sweet and gentle and sensitive man
With an obsessive nature and deep fascination
for numbers
And a complete infatuation
with the calculation of Pi

Chorus: Oh he love, he love, he love
He does love his numbers
And they run, they run, they run him
In a great big circle
In a circle of infinity

“a sentimental ode to a mathematician, audacious in both subject matter and treatment. The chorus is the number sung to many, many decimal places.” [150 – wrong after 50] — Observer Review
2002. Kanada computed $\pi$ to over 1.24 trillion decimal digits. His team first computed $\pi$ in hex (base 16) to 1,030,700,000,000 places, using good old Machin type relations:

\[
\pi = 48 \tan^{-1} \frac{1}{49} + 128 \tan^{-1} \frac{1}{57} - 20 \tan^{-1} \frac{1}{239} \\
+ 48 \tan^{-1} \frac{1}{110443} \quad \text{(Takano, pop-song writer 1982)}
\]

\[
\pi = 176 \tan^{-1} \frac{1}{57} + 28 \tan^{-1} \frac{1}{239} - 48 \tan^{-1} \frac{1}{682} \\
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Yasumasa Kanada

3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117068

11.00100100001111110110101000100010110100011000100011010011000100110001100100010110010111100

- The decimal expansion was checked by converting it back to hex.
  - Base conversion require pretty massive computation.

- Six times as many digits as before: hex and decimal ran 600 hrs on same 64-node Hitachi — at roughly 1 T flop/sec (2002).

- 2002 hex-pi computation record broken 3 times in 2009 — quite spectacularly. We will see that:

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J.M. Borwein  Life of Pi (CARMA)
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3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986628034825342117068

\[ \begin{array}{c}
3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986628034825342117068
\\[10pt]
\end{array} \]

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This took 131 days but he only used a single 4-core workstation with a lot of storage and even more human intelligence!

- For full details of this feat and of Takahashi’s most recent computation one can look at Wikipedia /wiki/Chronology_of_computation_of_pi
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Two New Pi Guys: Alex Yee and his Elephant

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There is probably no number in mathematics (with the possible exception of $e$) that is more celebrated than the one equal to the ratio of a circle’s circumference to its diameter. This number is denoted by the Greek letter $\pi$ ($\pi$). $\pi$ is approximately equal to 3.14159, but its decimal representation neither ends nor settles into a repeating pattern. In fact, on Oct. 16, 2013, Alexander J. Yee and Shigeki Kondo completed the task of using a custom-built computer (shown in Fig. 1) for 371 days, to calculate $\pi$ to 10 trillion digits!

To appreciate this accuracy, let me note that if we wanted to express the radius of the observable universe in terms of the radius of the hydrogen atom, about 40 digits would have sufficed.

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Figure 1. The computer used by Alexander Yee and Shigeki Kondo to calculate $\pi$ to 10 trillion digits (reproduced by permission from Alexander Yee)
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1990. Rabinowitz and Wagon found a ‘spigot’ algorithm for $\pi$: It ‘drips’ individual digits (of $\pi$ in any desired base) using all previous digits.

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What BBP Does?

Prior to **1996**, most folks thought to compute the \(d\)-th digit of \(\pi\), you had to generate the (order of) the entire first \(d\) digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of \(\pi\). In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of \(\pi\). It produces:

  - a modest-length string hex or binary digits of \(\pi\), beginning at an any position, *using no prior bits*;
    - is implementable on any modern computer;
    - requires no multiple precision software;
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    - a computational cost growing only slightly faster than the digit position.
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What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$
\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i + 1} - \frac{2}{8i + 4} - \frac{1}{8i + 5} - \frac{1}{8i + 6} \right) \quad (16)
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- The millionth hex digit (four millionth binary digit) of $\pi$ can be found in under 30 secs on a fairly new computer in Maple (not C++) and the billionth in 10 hrs.

Equation (16) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

$$
\pi = 4 \, _2F_1 \left( 1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{2} \right) - \log 5
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Proof of (16). For $0 < k < 8$,

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\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1 - x^8} \, dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^\infty x^{k-1+8i} \, dx = \frac{1}{2^{k/2}} \sum_{i=0}^\infty \frac{1}{16^i(8i + k)}.
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Thus, one can write

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Tuning BBP Computation

- **1997.** Fabrice Bellard of INRIA computed 152 bits of $\pi$ starting at the trillionth position;
  - in 12 days on 20 workstations working in parallel over the Internet.

Bellard used the following variant of (16):

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(2k+1)} - \frac{1}{64} \sum_{k=0}^{\infty} \frac{(-1)^k}{1024^k} \left( \frac{32}{4k+1} + \frac{8}{4k+2} + \frac{1}{4k+3} \right)$$  \hspace{1cm} (17)

This frequently-used formula is a little faster than (16).

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Hexadecimal Digits

1998. Colin Percival, a 17-year-old at Simon Fraser, found the five trillionth and ten trillionth hex digits on 25 machines.

2000. He then found the quadrillionth binary digit is 0.

- He used 250 CPU-years, on 1734 machines in 56 countries.
- The largest calculation ever done before Toy Story Two.

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<tr>
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<td>ECB840E21926EC</td>
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<tr>
<td>$10^9$</td>
<td>85895585A0428B</td>
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Abstract

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August 27, 2012 Ed Karrel found 25 hex digits of π **starting after** the 10^{15} position

- They are 353CB3F7F0C9ACCFA9AA215F2
- Using **BBP** on **CUDA** (too ‘hard’ for Blue Gene)
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See www.karrels.org/pi/,
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BBP Formulas Explained

Base-$b$ BBP numbers are constants of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)b^k},$$

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where $p(k)$ and $q(k)$ are integer polynomials and $b = 2, 3, \ldots$.

- I illustrate why this works in binary for $\log 2$. We start with:

$$\log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k}$$

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as discovered by Euler.

- We wish to compute digits beginning at position $d + 1$.

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- The key: the numerator in (20), $2^{d-k} \mod k$, can be found rapidly by binary exponentiation, performed modulo $k$. So,

$$3^{17} = (((((3^2)^2)^2)^2) \cdot 3$$

uses only 5 multiplications, not the usual 16. Moreover, $3^{17}$ mod 10 is done as $3^2 = 9; 9^2 = 1; 1^2 = 1; 1^2 = 1; 1 \times 3 = 3$. 

CARMA
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Catalan’s Constant $G$: and BBP for $G$ in Binary

The simplest number not proven irrational is

$$G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots, \quad \frac{\pi^2}{12} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

2009. $G$ is calculated to 31.026 billion digits. Records often use:

$$G = \frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{(2n)(2n+1)^2} + \frac{\pi}{8} \log(2 + \sqrt{3}) \quad \text{(Ramanujan)} \quad (21)$$

- holds since $G = -T\left(\frac{\pi}{4}\right) = -\frac{3}{2} T\left(\frac{\pi}{12}\right)$ where $T(\theta) := \int_0^\theta \log \tan \sigma d\sigma$.

- An 18 term binary BBP formula for $G = 0.9159655941772190 \ldots$ is:

$$G = \sum_{k=0}^{\infty} \frac{1}{4^{6k+3}} \left( \frac{3072}{(24k+1)^2} - \frac{3072}{(24k+2)^2} - \frac{23040}{(24k+3)^2} + \frac{12288}{(24k+4)^2} - \frac{768}{(24k+5)^2} + \frac{9216}{(24k+6)^2} + \frac{10368}{(24k+8)^2} + \frac{2496}{(24k+9)^2} - \frac{192}{(24k+10)^2} \right)$$

Eugene Catalan (1818-94) – a revolutionary
Catalan’s Constant $G$: and BBP for $G$ in Binary

The simplest number not proven irrational is

$$G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots, \quad \frac{\pi^2}{12} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

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2009. $G$ is calculated to 31.026 billion digits. Records often use:

$$G = \frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{(2n)!(2n+1)^2} + \frac{\pi}{8} \log(2 + \sqrt{3}) \quad \text{(Ramanujan)} \quad (21)$$

– holds since $G = -T\left(\frac{\pi}{4}\right) = -\frac{3}{2} T\left(\frac{\pi}{12}\right)$ where $T(\theta) := \int_{0}^{\theta} \log \tan \sigma d\sigma$.

An 18 term binary BBP formula for $G = 0.9159655941772190\ldots$ is:

$$G = \sum_{k=0}^{\infty} \frac{1}{4^k \cdot 4 + 6} \left( \frac{3072}{(24k+1)^2} - \frac{23040}{(24k+3)^2} + \frac{12288}{(24k+4)^2} \right)$$

Eugene Catalan (1818-94)– a revolutionary
Catalan’s Constant $G$: and BBP for $G$ in Binary

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\[
G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots, \quad \frac{\pi^2}{12} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots
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An 18 term binary BBP formula for $G = 0.9159655941772190\ldots$ is:

![Binary BBP Formula for G](image)

Eugene Catalan (1818-94)– a revolutionary

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2009. $G$ is calculated to 31.026 billion digits. Records often use:

$$G = \frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} + \frac{\pi}{8} \log(2 + \sqrt{3}) \quad \text{(Ramanujan)} \quad (21)$$

holds since $G = -T\left(\frac{\pi}{4}\right) = -\frac{3}{2} T\left(\frac{\pi}{12}\right)$ where $T(\theta) := \int_0^\theta \log \tan \sigma d\sigma$.

– An 18 term binary BBP formula for $G = 0.9159655941772190\ldots$ is:

$$G = \sum_{k=0}^{\infty} \frac{1}{4^k+6} + \frac{1}{10} + \frac{1}{8} + \frac{1}{18} + \cdots$$

$$G = \sum_{k=0}^{\infty} \frac{1}{4^k+6} \left( \frac{3072}{(24k+1)^2} - \frac{3072}{(24k+2)^2} + \frac{23040}{(24k+3)^2} - \frac{12288}{(24k+4)^2} \right)$$

$$- \frac{768}{(24k+5)^2} + \frac{9216}{(24k+6)^2} + \frac{10368}{(24k+7)^2} + \frac{2496}{(24k+9)^2} - \frac{648}{(24k+10)^2}$$

$$+ \frac{12}{(24k+12)^2} - \frac{168}{(24k+13)^2} + \frac{48}{(24k+15)^2} + \frac{39}{(24k+16)^2}$$

Eugene Catalan (1818-94)– a revolutionary
A Better Formula for $G$

A 16 term formula in concise BBP notation is:

$$G = P(2, 4096, 24, \vec{v}) \quad \text{where}$$

$$\vec{v} := (6144, -6144, -6144, 0, -1536, -3072, -768, 0, -768, -384, 192, 0, -96, 96, 96, 0, 24, 48, 12, 0, 12, 6, -3, 0)$$

It takes almost exactly $\frac{8}{9}$th the time of 18 term formula for $G$.

- This makes for a very cool calculation
- Since we can not prove $G$ is irrational, Who can say what might turn up?
What About **Base Ten?**

- The first integer logarithm with no known binary BBP formula is $\log 23$ (since $23 \times 89 = 2^{10} - 1$).

Searches conducted by numerous researchers for base-ten formulas have been unfruitful. Indeed:

2004. D. Borwein (my father), W. Gallway and I showed there are no BBP formulas of the Machin-type of (16) for $\pi$ if base is not a power of **two**.

- Bailey and Crandall have shown connections between the existence of a $b$-ary BBP formula for $\alpha$ and its base $b$ **normality** (via a dynamical system conjecture).
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Pi Photo-shopped: a 2010 PiDay Contest

“Noli Credere Pictis”
Thanks to Dave Broadhurst, a ternary BBP formula exists for $\pi^2$ (unlike $\pi$):

$$\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \times \left\{ \frac{243}{(12k + 1)^2} - \frac{405}{(12k + 2)^2} - \frac{81}{(12k + 4)^2} \right\}$$

$$- \frac{27}{(12k + 5)^2} - \frac{72}{(12k + 6)^2} - \frac{9}{(12k + 7)^2}$$

$$- \frac{9}{(12k + 8)^2} - \frac{5}{(12k + 10)^2} + \frac{1}{(12k + 11)^2} \right\}$$
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A Partner Binary BBP Formula for $\pi^2$

$$\pi^2 = \frac{9}{8} \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left\{ \frac{16}{(6k + 1)^2} - \frac{24}{(6k + 2)^2} - \frac{8}{(6k + 3)^2} - \frac{6}{(6k + 4)^2} + \frac{1}{(6k + 5)^2} \right\}$$

- We do not fully understand why $\pi^2$ allows BBP formulas in two distinct bases.
- $4\pi^2$ is the area of a sphere in three-space (L).
- $\frac{1}{2}\pi^2$ is the volume inside a sphere in four-space (R).
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IBM’s New Record Results

Algorithm (What We Did)

Dave Bailey, Andrew Mattingly (L) and Glenn Wightwick (R) of IBM Australia, and I, have obtained and (nearly) confirmed:

1. 106 digits of $\pi^2$ base 2 at the ten trillionth place base 64
2. 94 digits of $\pi^2$ base 3 at the ten trillionth place base 729
3. 150 digits of $G$ base 2 at the ten trillionth place base 4096

on a 4-rack BlueGene/P system at IBM’s Benchmarking Centre in Rochester, Minn, USA.
An enormous amount of delicate computation: **1380 years** is a long time. Suppose a spanking new IBM single-core PC went back **1381 years**.

- It would find itself in **632 CE**.
- The year that Mohammed died, and the Caliphate was established. If it then calculated $\pi$ nonstop:
  - Through the Crusades, black plague, Moguls, Renaissance, discovery of America, Gutenberg, Reformation, invention of steam, Napoleon, electricity, WW2, the transistor, fiber optics,...
- With no breaks or break-downs:
- It would have finished in **2012**.

August 2013, *Notices of the AMS*
The 3 Records Use Over 1380 CPU Years (135 rack days)

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\begin{itemize}
  \item August 2013, Notices of the AMS
  \begin{itemize}
    \item http://www.ams.org/notices/201307/rnoti-p844.pdf
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J.M. Borwein  Life of Pi (CARMA)
Algorithm (10 trillionth digits of $\pi^2$ in base 64 — in 230 years)

1. The calculation took, on average, 253529 seconds per thread. It was broken into 7 “partitions” of 2048 threads each. For a total of $7 \cdot 2048 \cdot 253529 = 3.6 \cdot 10^9$ CPU seconds.

2. On a single Blue Gene/P CPU it would take 115 years!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used
   $$\frac{7 \cdot 2048 \cdot 253529}{4096 \cdot 60 \cdot 60 \cdot 24} = 10.3 \text{ “rack days”}.$$

3. The verification run took the same time (within a few minutes): 106 base 2 digits are in agreement.

base-8 digits = 75|60114505303236475724500005743262754530363052416350634|573227604
60114505303236475724500005743262754530363052416350634|22021056612
Algorithm (10 trillionth digits of $\pi^2$ in base 729 — in 414 years)

1. The calculation took, on average, 795773 seconds per thread. It was broken into 4 “partitions” of 2048 threads each. For a total of $4 \cdot 2048 \cdot 795773 = 6.5 \cdot 10^9$ CPU seconds.

2. On a single Blue Gene/P CPU it would take 207 years!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used $\frac{4 \cdot 2048 \cdot 795773}{4096 \cdot 60 \cdot 60 \cdot 24} = 18.4$ “rack days”.

3. The verification run took the same time (within a few minutes): 94 base 3 digits are in agreement.

base-9 digits = 001|12264485064548583177111135210162856048323453468|10565567|635862
12264485064548583177111135210162856048323453468|04744867|134524345
Algorithm (10 trillionth digits of \( G \) in base 4096 — in 735 years)

1. The calculation took, on average, 707857 seconds per thread. It was broken into 8 “partitions” of 2048 threads each. For a total of \( 8 \cdot 2048 \cdot 707857 = 1.2 \cdot 10^{10} \) CPU seconds.

2. On a single Blue Gene/P CPU it would take 368 years!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used \( \frac{8 \cdot 2048 \cdot 707857}{4096 \cdot 60 \cdot 60 \cdot 24} = 32.8 \) “rack days”.

3. The verification run will take the same time (within a few minutes): xxx base 2 digits will be in agreement.

base-8 digits = 0176|347050513714777051122613371620125573272173245226000177545727
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx|xxxx
Thank You, One and All, and Happy Birthday, Albert

138. Links and References

1. The Pi Digit site: http://carma.newcastle.edu.au/bbp
2. Dave Bailey’s Pi Resources: http://crd.lbl.gov/~dhbailey/pi/

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6. Jonathan M. Borwein and Peter B. Borwein, Selected Writings on Experimental and Computational Mathematics, PsiPress. October 2010.4

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J.M. Borwein Life of Pi (CARMA)