ON HECKE'S THEOREM ON THE REAL ZEROS OF THE
L-FUNCTIONS AND THE CLASS NUMBER OF QUADRATIC
FIELDS

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Prof. Mordell, by generalizing a method of Deuring†, proves‡ that
Riemann’s hypothesis for $\zeta(s)$ is true if an infinity of different imaginary
quadratic fields $K(\sqrt{-d})$ exists with the same value of the class number
$h(-d)$. The proof depends on an asymptotic formula for $\zeta(s)L_d(s)$ for
d $\to \infty$, where $L_d(s)$ denotes the $L$-series belonging to the field.

I find that the same asymptotic formula may be applied to the study of
$L_d(s)$ instead of $\zeta(s)$, and that it leads to relations between the class number
$h(-d)$ and the real zeros of $L_d(s)$. The following result, including that due
to Hecke§, will be proved in this note:

Suppose that

$$a_n x^2 + b_n xy + c_n y^2 \quad (n = 1, 2, \ldots, h)$$

is the system of all reduced positive quadratic forms of fundamental
discriminant $-d = b_n^2 - 4a_n c_n < 0$, and that $h'$ denotes the quotient

$$h' = \frac{h}{\sum_{n=1}^{h} \frac{1}{a_n}}.$$
Then, when \( d \) is sufficiently large, corresponding to given constants \( \gamma > 0, \ g > 0 \), there exist constants \( \Gamma = \Gamma(\gamma) > 0, \ G = G(g) > 0 \), such that if \( L_d(s) \) has at least one zero in the interval
\[
1 - \frac{\gamma}{\log d} \leq s \leq 1,
\]
then
\[
h' \leq \Gamma \frac{\sqrt{d}}{\log d};
\]
and if \( L_d(s) \) has no zero in the interval
\[
1 - \frac{g}{\log d} \leq s \leq 1,
\]
then
\[
h' \geq G \frac{\sqrt{d}}{\log d}.\]

1. Suppose that
\[
Q = ax^2 + bxy + cy^2
\]
is a reduced quadratic form with integer coefficients and negative discriminant
\[
-d = b^2 - 4ac < 0,
\]
so that
\[
0 < a \leq \sqrt{\left(\frac{1}{4}d\right)}.
\]
From Mordell’s paper, formulae (4), (11), for \( d < -4 \),
\[
2\zeta_Q(s) = \sum_{x = -\infty}^{+\infty} \sum_{y = -\infty}^{+\infty} Q^{-s} \left( x^2 + y^2 > 0 \right)
\]
\[
= 2a^{-s} \zeta(2s) + 2d^{1-s} a^{s-1} \left( \frac{\Gamma(2s-1) \Gamma(s-\frac{1}{2}) \sqrt{\pi}}{\Gamma(s)} + O(1) \right)
\]
uniformly in the interval
\[
\frac{1}{2} \leq s \leq 1
\]
as \( d \to \infty \).

The zeta function \( \zeta_d(s) \) of the imaginary quadratic field \( K(\sqrt{-d}) \) satisfies the equations
\[
\zeta_d(s) = \zeta(s) L_d(s) = \sum_Q \zeta_Q(s),
\]
where
\[
L_d(s) = \sum_{n=1}^{\infty} \left( \frac{-d}{n} \right) n^{-s}
\]
is the corresponding \( L \)-series, and the summation in \( Q \) refers to all different
reduced forms $Q$. Hence

$$
\zeta(s) L_d(s) = f_d(s) \zeta(2s) + d^{s-\frac{1}{2}} f_d(1-s) \left( \frac{\zeta(2s-1) \Gamma(s-\frac{1}{2}) \sqrt{\pi}}{\Gamma(s)} + a_d(s) \right),
$$

where $f_d(s)$ is the finite Dirichlet series

$$
f_d(s) = \sum_q a_q^{-s},
$$

and where $a_d(s)$ denotes a real function of $s$, which is uniformly bounded in the interval

$$
\frac{1}{2} \leq s \leq 1
$$

when $d$ tends to infinity.

Now take an arbitrary constant $\sigma_0$ with

$$
\frac{1}{2} < \sigma_0 < 1.
$$

Then there exist five positive numbers $c_1, \ldots, c_5$, which depend only on $\sigma_0$, such that, for sufficiently large $d$,

$$
c_1 \leq \zeta(2s) \leq c_2, \quad c_3 \leq \frac{\zeta(2s-1) \Gamma(s-\frac{1}{2}) \sqrt{\pi(s-1)}}{\Gamma(s)} \leq c_4, \quad |a_d(s)| \leq c_5,
$$

uniformly in $s$ in the interval

$$
\sigma_0 \leq s \leq 1.
$$

Hence, if $\sigma_1$ denotes the number

$$
\sigma_1 = \max \left( \sigma_0, 1 - \frac{c_3}{2c_5} \right),
$$

the inequality

$$
c_6 = \frac{c_3}{2} \leq \left( \frac{\zeta(2s-1) \Gamma(s-\frac{1}{2}) \sqrt{\pi}}{\Gamma(s)} + a_d(s) \right) (s-1) \leq c_4 + \frac{c_3}{2} = c_7
$$

holds uniformly for $s$ in

$$
\sigma_1 \leq s \leq 1,
$$

as $d$ tends to infinity. We write

$$
A_d(s) = \zeta(2s), \quad B_d(s) = \left( \frac{\zeta(2s-1) \Gamma(s-\frac{1}{2}) \sqrt{\pi}}{\Gamma(s)} + a_d(s) \right) (s-1),
$$

and then have the result:

"There exist a constant $\sigma_1$, with $\frac{1}{2} < \sigma_1 < 1$, and also four positive constants $c_1, c_2, c_6, c_7$, such that

$$
\zeta(s) L_d(s) = A_d(s) f_d(s) - \frac{B_d(s)}{1-s} d^{s-\frac{1}{2}} f_d(1-s),
$$

..."
where \[ c_1 \leq A_d(s) \leq c_2, \quad c_6 \leq B_d(s) \leq c_7 \]
uniformly in \( s \) for the interval \( J \) or
\[ \sigma_1 \leq s \leq 1, \]
for all sufficiently large positive integers \( d \).

2. We write \( h = f_d(0), \quad \eta = f_d(1) \).
Since \( 0 < \alpha \leq \sqrt{\frac{1}{3} d} \leq \sqrt{d} \)
for reduced forms, we have obviously
\[ hd^{-\frac{1}{2}}(1-s) \leq f_d(1-s) \leq h, \]
and
\[ \eta \leq f_d(s) \leq \eta d^{\frac{1}{2}}(1-s). \]
Therefore
\[ h' d^{-(1-s)} \leq \frac{f_d(1-s)}{f_d(s)} \leq h', \]
where \( h' = h/\eta \).

Hence the two functions
\[ X_d(s) = \frac{(1-s) \zeta(s) L_d(s)}{B_d(s) f_d(s)}, \quad Y_d(s) = \frac{(1-s) \zeta(s) L_d(s)}{B_d(s) f_d(s) d^{1-s}}, \]
which in \( J \) have the same sign as \( \zeta(s) L_d(s) \) and hence the opposite sign to \( L_d(s) \), satisfy the inequalities

1. \[ X_d(s) = \frac{A_d(s)}{B_d(s)} (1-s) - \frac{f_d(1-s)}{f_d(s)} d^{1-s} \leq \frac{c_2}{c_6} (1-s) - \frac{h'}{\sqrt{d}} \]
and

2. \[ Y_d(s) = \frac{A_d(s)}{B_d(s)} (1-s) d^{-(1-s)} - \frac{f_d(1-s)}{f_d(s)} d^{-1} \geq \frac{c_1}{c_7} (1-s) d^{-(1-s)} - \frac{h'}{\sqrt{d}}. \]

When \( s \) lies in the interval \( J \) and is sufficiently near to \( s = 1 \), then obviously both functions are negative.

3. It is clear from (1) that \( -X_d(s) \), and therefore also \( L_d(s) \), is always positive in the interval
\[ \max \left( \sigma_1, 1 - \frac{c_6}{c_2} \frac{h'}{\sqrt{d}} \right) < s \leq 1. \]

We obtain as a special case

**Theorem 1.** Suppose that \( \gamma \) is a positive constant and that the integer \( d > 0 \) is greater than a certain number \( d_0 \) which depends only on \( \gamma \). Then,
if the $L$-series $L_d(s)$ has at least one zero in the interval

$$1 - \frac{\gamma}{\log d} \leq s \leq 1,$$

there exists a number $\Gamma > 0$, depending only on $\gamma$, such that

$$h' \leq \Gamma \frac{\sqrt{d}}{\log d}.$$ 

Next, in the inequality (2), the term

$$\frac{c_1}{c_7} (1-s) d^{-(1-s)}$$

has its maximum at $s = 1 - (\log d)^{-1}$, and is then equal to

$$\frac{c_1}{c_7} \frac{e^{-1}}{\log d};$$

in the interval

$$1 - \frac{1}{\log d} \leq s \leq 1$$

it assumes every value between this maximum and zero. Hence, when

$$h' \leq G \frac{\sqrt{d}}{\log d},$$

where $G$ is a positive number with

$$G < \frac{c_1}{c_7} \frac{e^{-1}}{\log d},$$

there exists a second positive number $g$ with $g \leq 1$, such that $Y_d(s)$, and so also $L_d(s)$, changes its sign at least once in the interval

$$1 - \frac{g}{\log d} \leq s \leq 1.$$

This result proves

**Theorem 2.** Suppose that $g$ is a positive constant and that the integer $d > 0$ is greater than a certain number $d_1$ which depends only on $g$. Then, if the $L$-series $L_d(s)$ has no zero in the interval

$$1 - \frac{g}{\log d} \leq s \leq 1,$$

there exists a number $G > 0$, depending only on $g$, such that

$$h' \geq G \frac{\sqrt{d}}{\log d}.$$