NOTE ON HYPOTHESIS $K$ OF HARDY AND LITTLEWOOD

KURT MAHLER*.


1. Some weeks ago, Prof. Mordell asked me in a letter whether the Diophantine equation

\[ x^3 + y^3 + z^3 = 1 \]

has an infinity of integer solutions other than the trivial ones such as $x = 1, y = -z$. I found the following answer to this question.

As is well known†,

\[ x^3 + y^3 + z^3 = u^3 \]

identically, if

\[ x = \rho^2 - \sigma \rho', \quad y = \sigma' \rho' - \rho^2, \quad z = \rho'^2 - \rho \sigma', \quad u = \rho'^2 - \rho \sigma, \]

where

\[ \rho = f^2 + 3g^2, \quad \rho' = f'^2 + 3g'^2, \quad \sigma = ff' + 3gg' + 3f'g - 3f'g', \quad \sigma' = ff' + 3gg' - 3f'g + 3f'g. \]

Here, obviously, $u = 1$, if $\rho' = 1$ and $\sigma = 0$; and this requires $f' = 1$, $g' = 0$, $f - 3g = 0$, whence $\rho = 12g^2$, $\sigma' = 6g$. Then

\[ x = 12^2 g^4, \quad y = 6g - 12^2 g^4, \quad z = 1 - 6.12g^3, \quad u = 1, \]

or, with $2g = \xi$,

\[ (9\xi^4)^3 + (3\xi - 9\xi^4)^3 + (1 - 9\xi^3)^3 = 1. \]

Thus, for every integer $\xi$, we find a non-trivial solution of (1).

* Received 10 December, 1935; read 12 December, 1935.
† L. E. Dickson, History of the theory of numbers, 2, 555.
More generally,
\[(9d^3 \xi^4) + (3d \xi - 9d^3 \xi^4) + d(1 - 9d^2 \xi^3)^3 = d,\]
and therefore both the equations
\[x^3 + y^3 + dz^3 = d, \quad d^2(x^3 + y^3) + z^3 = 1\]
have an infinity of integer solutions. Since
\[(6d^2 \xi^3 + 1)^3 + (-6d^2 \xi^3 + 1)^3 + 6d(-6d^2 \xi^2)^3 = 2\]
identically in $\xi$, there are also infinitely many integers $x, y, z$ for which
\[x^3 + y^3 + dz^3 = 2,\]
or
\[d^2(x^3 + y^3) + z^3 = d^2.\]
This last identity is a special case of
\[\lambda_1(\xi^n + a_1)^n + \lambda_2(\xi^n + a_2)^n + \ldots + \lambda_n(\xi^n + a_n)^n + \lambda_n(\xi^2)^n = \mu.\]
Here $n \geq 3, a_1, a_2, \ldots, a_{n-1}$ are $n-1$ different integers with $a_1 + \ldots + a_{n-1} = 0$, and
\[\lambda_v = (-\lambda)^{v-\lambda} \prod_{\lambda = 1}^{\lambda-1} \prod_{\kappa = 1}^{\kappa-1} (a_\kappa - a_\lambda) \quad (v = 1, 2, \ldots, n-1);\]
\[\lambda_n = -\left(\frac{n}{2}\right) \prod_{\lambda = 1}^{\lambda-1} \prod_{\kappa = 1}^{\kappa-1} (a_\kappa - a_\lambda); \quad \mu = \sum_{v=1}^{n-1} a_v^n \lambda_v.\]
Hence, when $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu$ have these values, there are an infinity of integers $x_1, x_2, \ldots, x_n$ for which
\[\lambda_1 x_1^n + \lambda_2 x_2^n + \ldots + \lambda_n x_n^n = \mu.\]

2. About ten years ago, in a paper on Waring's problem, Hardy and Littlewood* gave consequences of the following

**Hypothesis K.** If $n \geq 2$ is an integer, then the number of solutions of
\[x_1^n + x_2^n + \ldots + x_n^n = N,\]
in non-negative integers $x_1, x_2, \ldots, x_n$, is $O(N^c)$ for every positive $c$ and large $N$.

The hypothesis is known to be true for $n = 2$, but the number of solutions is not bounded, and indeed is sometimes larger than
\[\exp \left( c \frac{\log N}{\log \log N} \right),\]

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where $c > 0$ is an appropriate constant. Recent theorems of Chowla* and Erdős† show that these last results are also true for larger values of $n$; but the truth or falsity of the hypothesis itself, for $n \geq 3$, has remained undecided.

I can now prove that Hypothesis $K$ is false for $n = 3$. If we replace $\xi$ by $\xi/\eta$ in (2), it becomes

$$ (g\xi^4)^3 + (3\xi\eta^3 - 9\xi^4)^3 + (\eta^4 - 9\xi^3 \eta)^3 = \eta^{12}, $$

and here all three cubes will be positive for

$$ \eta > 0, \quad 0 < \xi < 9^{-\frac{3}{4}} \eta. $$

Hence, for all large $N$ which are 12th powers, the equation

$$ x^3 + y^3 + z^3 = N \quad (N = \eta^{12}) $$

has at least

$$ 9^{-\frac{3}{4}} N^{\frac{1}{3}} $$
solutions in non-negative integers.

An analogous result holds for the more general equations

$$ x^3 + y^3 + dz^3 = N, \quad d^2(x^3 + y^3) + z^3 = N \quad (d = 1, 2, 3, \ldots), $$
as follows in the same way from (3). The identities (4) and (5) are of smaller value, since the terms of their left-hand sides are not all of the same sign. They only lead to the following result. Suppose that $n \geq 3$. Then there are integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and positive constants $A_1, A_2, \ldots, A_n, C$ such that

$$ \lambda_1 x_1^n + \lambda_2 x_2^n + \ldots + \lambda_n x_n^n = N, $$

$$ |x_1^n| < A_1 N, \quad |x_2^n| < A_2 N, \quad \ldots, \quad |x_n^n| < A_n N $$

for an infinity of $N$ and more than

$$ CN^{n-2} $$

sets of integers $x_1, x_2, \ldots, x_n$. This result suggests that Hypothesis $K$ is probably false generally for $n \geq 3$.

Mathematical Department,
The University,
Groningen (Holland).

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* Indian Physico-Mathematical Journal, 6 (1935), 65–68.