On the fractional parts of the powers of a rational number.

By

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Let \( u \) and \( v \) be two coprime integers with \( u > v > 1 \), such that \( \frac{u}{v} > 1 \), suppose that

\[
\varphi_n = \left( \frac{u}{v} \right)^n - \left[ \left( \frac{u}{v} \right)^n \right].
\]

Then the following results, as special cases of more general theorems, are proved in this paper:

a: \[
\lim_{n \to \infty} vn^\varphi_n = \infty.
\]

b: When \( \varepsilon \) is a positive constant and

\[
\varphi_n \leqslant u^{-\varepsilon n}
\]

for an infinite sequence of positive integers \( n = n_1, n_2, n_3, \ldots \) with \( n_{\nu+1} > n_\nu \), then

\[
\limsup_{n \to \infty} \frac{n_\nu + 1}{n_\nu} = \infty.
\]

The proofs of a) and b) depend on generalizations of the Thue-Siegel theorem, due to Schneider or myself, and are very simple.

I.

1) Some years ago, I proved the following theorem 1):

LEMMA 1: Let $F(x, y)$ be an irreducible binary form of degree $n \geq 3$ with integer coefficients, $x$ and $y$ two coprime integers, $P_1, P_2, \ldots, P_t$ ($t \geq 1$) a finite number of different prime numbers, and $Q(x, y) = P_1^{h_1} P_2^{h_2} \ldots P_t^{h_t}$ the greatest product of powers of these primes, which divides $F(x, y)$. Then

$$Q(x, y) \leq c_0 \max (|x|, |y|)^{2\sqrt{n}},$$

where $c_0 > 0$ is a constant, which does not depend on $x$ and $y$.

From this lemma, the following one is a trivial consequence:

LEMMA 2: Let $a, b, x$ be three non-vanishing integers, $n \geq 5$ a prime number, $v$ an integer $\geq 2$, and $q(x) = v^t$ the highest power of $v$, which divides $ax^n - b$. Then

$$q(x) \leq c_1 |x|^{2\sqrt{n}} + 1,$$

where $c_1 > 0$ is a constant, which does not depend on $x$.

Proof: Since $n$ is an odd prime, the binary form $F(x, y) = ax^n - by^n$ either is irreducible, or of the form

$$F(x, y) = (a x - b y) G(x, y),$$

where $a, b$ are integers, and $G(x, y)$ is an irreducible binary form of degree $n - 1$. Suppose that $P_1, P_2, \ldots, P_t$ are the different prime factors of $v$. Then apply Lemma 1 with $y = 1$ to $F(x, y)$ in the first case, and to $G(x, y)$ in the second case. Then we get

$$q(x) = O(|x|^{2\sqrt{n}})$$
in the first case, and

$$q(x) = O(|x| \cdot |x|^{2\sqrt{n} - 1})$$
in the second case, since $ax - by = O(x)$.

THEOREM 1: Let $a, b, u, v$ be four non-vanishing integers with $u > v > 1$. Then the equation

$$(1): \quad a u^x - v^x y = b$$

has at most a finite number of solutions in integers $x \geq 0$ and $y$.

Proof: Let $\lambda$ be the number

$$\lambda = \frac{\log v}{\log u},$$

thus $0 < \lambda < 1$. Take for $n$ a prime number $\geq 5$, such that
1+2 \sqrt{n} \leq \lambda \cdot n;

this condition is satisfied, for instance, when

\[ n \geq \left( \frac{3}{\lambda} \right)^2. \]

Obviously, to every solution \( x, y \) of (1), there are two integers \( \xi \) and \( \nu \) with

\[ x = n \xi + \nu, \quad \xi \geq 0, \quad 0 \leq \nu \leq n - 1, \quad au^\nu (u^\xi)^n - b = v^\nu y (v^\xi)^n. \]

Hence

\[ au^\nu X^n - b, \quad \text{where} \quad X = u^\xi, \]

is divisible by a power of \( v \), which, at least, is equal to

\[(v^\xi)^n = X^{h_n}.

But by Lemma 2, applied to each of the \( n \) polynomials

\[ au^\nu X^n - b \quad (\nu = 0, 1, \ldots, n - 1), \]

this power of \( v \) must be

\[ O(X^{2\nu + 1}), \]

and therefore \( X \) and \( x \) cannot be arbitrarily large, i.e., (1) has at most a finite number of solutions. Q. E. D.

**THEOREM 2**: Under the conditions of theorem 1, the congruence

\[ au^x \equiv d \pmod{v^x} \]

can hold only for a finite number of integers \( x > 0 \).

**THEOREM 3**: Suppose that \( a, u, v \) are integers with \( a \neq 0, u > v > 1, v \neq u \). Then

\[ \lim_{n \to \infty} v^n \left\{ a \left( \frac{u}{v} \right)^n - \left[ a \left( \frac{u}{v} \right)^n \right] \right\} = \infty. \]

These two theorems are trivial consequences of Theorem 1. In the case of Theorem 3, the additional condition \( v \neq u \) makes it impossible, that \( au^n - v^n y = 0 \) has an infinity of solutions.

II.

2) The following theorem can be proved:

**LEMMA 3**: Let \( \theta \neq 0 \) be an algebraic number and \( p_1 | q_1, p_2 | q_2, p_3 | q_3, \ldots \)
... an infinite sequence of simplified fractions with the following properties:

\[ a: \quad 1 \leq q_1 < q_2 < q_3 \leq \ldots \]

\[ b: \quad \text{For every } n, p_n \text{ and } q_n \text{ can be written as} \]
\[ p_n = P_{1}^{h_{1}} \cdots P_{s}^{h_{s}} p_n^*, \quad q_n = Q_{1}^{k_{1}} \cdots Q_{t}^{k_{t}} q_n^*, \]

where \( P_{1}, \ldots, P_{s}, Q_{1}, \ldots, Q_{t} \) is a given finite system of different prime numbers, \( h_{1}, \ldots, h_{s}, k_{1}, \ldots, k_{t} \) are integers \( \geq 0 \) and \( p_n^*, q_n^* \) are integers, such that as \( n \to \infty \)
\[ p_n^* = O(p_n^\alpha), \quad q_n^* = O(q_n^\beta), \]

where \( \alpha, \beta \) are given constants with \( 0 \leq \alpha \leq 1, \quad 0 \leq \beta \leq 1. \)

\[ c: \quad \text{For every } n \]
\[ \left| \frac{\Theta - p_n}{q_n} \right| < q_n^{-\gamma}, \]

where \( \gamma \) is a constant with \( \gamma > \alpha + \beta. \)

Then
\[ \lim_{n \to \infty} \sup \frac{\log q_{n+1}}{\log q_n} = \infty. \]

For \( \alpha = \beta = 1, s = t = 0, \) this theorem was proved by Th. Schnei-
der,\(^2\), and by using his method, I proved it\(^3\) for \( \alpha = 0, \beta = 1, t = 0, \) or for \( \alpha = 1, \beta = 0, s = 0. \) The same method, however, leads also to the general result of Lemma 3, as a study of the proof shows. (It is sufficient for this purpose, to use approximation polynomials of the form
\[ R(z_1, z_2, \ldots, z_n) = \sum R_{l_1, l_2, \ldots, l_k} z_1^{l_1} z_2^{l_2} \cdots z_k^{l_k}, \]
where the summation sign refers to all integers \( l_1, l_2, \ldots, l_k \) with
\[ 0 \leq l_1 \leq r_1, \quad 0 \leq l_2 \leq r_2, \ldots, \quad 0 \leq l_k \leq r_k, \quad \frac{k}{2} \leq 1 - \varepsilon \leq 2, \quad \sum_{\varepsilon = 1}^{k} \frac{l_{\varepsilon}}{r_{\varepsilon}} \leq \frac{k}{2} (1 + \varepsilon). \]

Compare Kapitel 1 of my paper, in particular § 6 and § 8).

\(^2\) Journal reine u. angew. Math. 175 (1937), "Über die Approximation algebraischer Zahlen".

\(^3\) Proceedings Royal Academy Amsterdam, 39 (1937), 633—640, 729—737.
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**THEOREM 4:** Suppose that \( \vartheta \neq 0 \) is an algebraic number and that \( u \) and \( v \) are integers with \( u > v > 1 \), \( v + u \), that \( \varepsilon \) is a positive constant, and that \( n = n_1, n_2, n_3, \ldots \) is an infinite increasing sequence of positive integers, for which

\[
\vartheta \left( \frac{u}{v} \right)^n - \left[ \vartheta \left( \frac{u}{v} \right)^n \right] \leq u - \varepsilon n.
\]

Then

\[
\limsup_{\gamma \to \infty} \frac{n_{\gamma + 1}}{n_\gamma} = \infty.
\]

**Proof:** If again

\[
\lambda = \frac{\log v}{\log u},
\]

then (2) obviously is equivalent to

\[
0 \leq \vartheta - \frac{v^n \left[ \vartheta \left( \frac{u}{v} \right)^n \right]}{u^n} \leq \left( \frac{v}{u} \right)^n u - \varepsilon n = u - (1 - \lambda + \varepsilon)n.
\]

Hence, Lemma 3 can be applied with

\[
p = v^n \left[ \vartheta \left( \frac{u}{v} \right)^n \right], \quad p^* = \left[ \vartheta \left( \frac{u}{v} \right)^n \right], \quad q = u^n, \quad q^* = 1,
\]

so that

\[
\alpha = 1 - \lambda, \quad \beta = 0, \quad \alpha + \beta < \gamma = 1 - \lambda + \varepsilon,
\]

and the assertion follows at once.

Probably, (2) has only a finite number of solutions for \( n \).

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