ON MINKOWSKI’S THEORY OF REDUCTION OF
POSITIVE DEFINITE QUADRATIC FORMS

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[Received 8 June 1938]

MINKOWSKI* called a positive definite quadratic form in \( n \) variables

\[
F(x) = \sum_{h,k=1}^{n} a_{hk} x_h x_k
\]

reduced, if, for \( h = 1, 2, \ldots, n \) and for all systems of \( n \) integers \( x_1, \ldots, x_n \),

\[
F(x) \geq a_{hh},
\]

when the greatest common divisor

\[
g.c.d. (x_h, x_{h+1}, \ldots, x_n) = 1,
\]

and if certain \( n-1 \) other unimportant inequalities were satisfied. He proved that, for reduced forms of discriminant \( D \),

\[
\lambda_n a_{11} a_{22} \ldots a_{nn} \leq D,
\]

where \( \lambda_n > 0 \) depends only on \( n \). L. Bieberbach and I. Schur† showed that

\[
\lambda_n \geq \left( \frac{48}{125} \right)^{\frac{1}{3}} (n^2 - n),
\]

and R. Remak‡ in a recent paper improved this to

\[
\lambda_n \geq \gamma_n \left( \frac{4}{3} \right)^{\frac{1}{3}(n-3)(n-4)}
\]

where \( \gamma_n \) is Hermite’s constant, for which \( D \geq \gamma_n a_{nn}^n \).

As Remak’s proof is rather long, I give here a very short and simple proof (which I had obtained before the paper of Remak appeared) for the slightly weaker inequality (since \( \frac{4}{9} < \frac{4}{3} \))

\[
\lambda_n \geq 2 - 2n \left( \frac{4}{9} \right)^{\frac{1}{3}(n-1)(n-2)} \frac{\{ \Gamma(\frac{1}{2}) \}^{2n}}{\{ \Gamma(1 + \frac{1}{2}n) \}^2}.
\]

My proof is valid for the reduction of arbitrary convex bodies; it employs Minkowski’s theorem on the successive minima of a convex body.||

‡ Compositio Math. 5 (1938), 368–91.

\[
\gamma_n \geq \frac{\pi^n}{2^n \Gamma(2 + \frac{1}{2}n)^2},
\]

due to Blichfeldt.

|| Geometric der Zahlen, 218.
1. Let \( f(x) = f(x_1, \ldots, x_n) \) be a real function of \( n \) real variables \( x_1, \ldots, x_n \) (\( n \geq 2 \)) with the following properties:

(i) \( f(0, \ldots, 0) = 0, \ f(x_1, \ldots, x_n) > 0 \) for \( \sum_{h=1}^{n} x_h^2 > 0 \);

(ii) \( f(tx_1, \ldots, tx_n) = |t|f(x_1, \ldots, x_n) \) for real \( t \);

(iii) \( f(x_1+y_1, \ldots, x_n+y_n) \leq f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n) \).

Then, for \( t > 0 \), the inequality \( f(x) \leq t \) defines a convex body \( K(t) \) in \( n \) dimensions, of volume \( J(t) = Jt^n \), where \( J \) denotes the volume of the body \( K(1) \), say \( K \).

For every \( t > 0 \), since \( K(t) \) contains only a finite number of lattice points, it is possible to apply Minkowski’s method of reduction* to the function \( f(x) \). Let \( M_h \) (\( h = 1, \ldots, n \)) be the set of all lattice points \( (x_1, \ldots, x_n) \) whose last \( n-h+1 \) coordinates \( x_h, x_{h+1}, \ldots, x_n \) are relatively prime, and let

\[
a_h = f(\delta_{h1}, \delta_{h2}, \ldots, \delta_{hn}) = f(\delta_h) \quad (h = 1, 2, \ldots, n),
\]

where \( \delta_{hk} \) is Kronecker’s symbol:

\[
\delta_{hh} = 1, \quad \text{but} \quad \delta_{hk} = 0 \quad \text{for} \quad h \neq k \quad (h, k = 1, 2, \ldots, n).
\]

**Definition.** The function \( f(x) \) and the corresponding convex body \( K \) are called ‘reduced’, if for each \( h = 1, \ldots, n \) and for all lattice points \( (x) \) in \( M_h \)

\[
f(x) \geq a_h.
\]

As in Minkowski’s paper, it is easily proved that \( f(x) \) can be reduced by applying a suitable unimodular linear transformation

\[
x_h \rightarrow \sum_{k=1}^{n} a_{hk}x_k \quad (h = 1, 2, \ldots, n)
\]

with integer coefficients.

2. **Theorem.** For reduced functions \( f(x) \)

\[
a_1a_2\ldots a_n \leq \frac{2^n (\frac{3}{2})^{\frac{1}{2}(n-1)(n-2)}}{J}.
\]

**Proof.** Minkowski† proved that there are \( n \) independent lattice points

\[
(p_h) = (p_{h1}, \ldots, p_{hn}) \quad (h = 1, 2, \ldots, n)
\]

such that, if

\[
S_h = f(p_h) = f(p_{h1}, \ldots, p_{hn}) \quad (h = 1, 2, \ldots, n),
\]

then

\[
S_1 \leq S_2 \leq \ldots \leq S_n, \quad S_1S_2\ldots S_n \leq \frac{2^n}{J^2},
\]

† Geometrie der Zahlen, 218.
and
\[ f(x) \geq S_h \]
for all lattice points \((x)\) which are linearly independent of \((p_1), (p_2), \ldots, (p_{h-1})\).

Obviously, \((p_1)\) belongs to \(M_1\); hence
\[ a_1 \leq S_1. \]  \(1\)
(More exactly \(a_1 = S_1\), but we do not need this.)

Suppose that we have already obtained \(m-1\) positive absolute constants\[ \gamma_1, \gamma_2, \ldots, \gamma_{m-1}, \]
such that\[ a_h \leq \gamma_h S_h \quad (h = 1, 2, \ldots, m-1). \]  \(2\)
By (1) in particular\[ \gamma_1 = 1, \]
and we now find a similar constant \(\gamma_m\) for which
\[ a_m \leq \gamma_m S_m. \]  \(3\)

The \(m\) lattice points \((p_1), \ldots, (p_m)\) are independent. Hence at least one of them, say \((p_i) = (p_{i1}, \ldots, p_{im})\) \((i = 1 \text{ or } 2 \text{ or } \ldots \text{ or } m)\), has its last \(n-m+1\) coordinates \(p_{i, m+1}, \ldots, p_{im}\) not all zero. Therefore the greatest common divisor
\[ \text{g.c.d.} \quad (p_{im}, p_{i, m+1}, \ldots, p_{in}) = d_m \neq 0. \]
If \(d_m = 1\), then \((p_i)\) belongs to \(M_m\), and therefore \(a_m \leq S_i\), i.e.
\[ a_m \leq S_m. \]  \(4\)
Suppose, however, that \(d_m \geq 2\). Then we can find \(m-1\) integers \(g_1, g_2, \ldots, g_{m-1}\), such that
\[ p_{ih} + g_h \equiv 0 \pmod{d_m} \quad \text{and} \quad |g_h| \leq \frac{1}{2} d_m \quad (h = 1, 2, \ldots, m-1). \]
Hence, writing the left-hand side in vector form,
\[ \left(\frac{p_{i1} + g_1}{d_m}, \ldots, \frac{p_{i, m-1} + g_{m-1}}{d_m}, \frac{p_{im}}{d_m}, \ldots, \frac{p_{in}}{d_m}\right) = \frac{1}{d_m} \left( \sum_{h=1}^{m-1} g_h (p_h + (p_i)) \right) \]
is a lattice point of the set \(M_m\). Therefore, from (ii) and (iii), since \(d_m \geq 2\),
\[ a_m \leq \frac{1}{d_m} \left( \sum_{h=1}^{m-1} |g_h| a_h + S_i \right) \leq \frac{1}{2} \left( \sum_{h=1}^{m-1} \gamma_h S_h + S_m \right), \]
i.e.
\[ a_m \leq \frac{\gamma_1 + \gamma_2 + \ldots + \gamma_{m-1} + 1}{2} S_m. \]  \(5\)
Put
\[ \gamma_m = \max \left(1, \frac{\gamma_1 + \gamma_2 + \ldots + \gamma_{m-1} + 1}{2} \right). \]
Then, from (4), (5), we see that (3) is satisfied.
Now
\[ \gamma_1 = 1, \quad \gamma_2 = \max \left( 1, \frac{1+1}{2} \right) = 1, \]
\[ \gamma_3 = \max \left( 1, \frac{1+1+1}{2} \right) = \frac{3}{2}, \]
\[ \gamma_4 = \max \left( 1, \frac{1+1+\frac{3}{2}+1}{2} \right) = \frac{9}{4} = \left( \frac{3}{2} \right)^2. \]

Suppose, then, that
\[ \gamma_h = \left( \frac{3}{2} \right)^{h-2} \quad \text{for} \quad h = 2, 3, \ldots, m-1. \] (6)
Then
\[ \gamma_m = \max \left( 1, \frac{3 + \left( \frac{3}{2} \right)^1 + \left( \frac{3}{2} \right)^2 + \ldots + \left( \frac{3}{2} \right)^{m-3}}{2} \right) = \frac{1}{2} \left( 3 + \frac{\left( \frac{3}{2} \right)^{m-2} - \left( \frac{3}{2} \right)^1}{\frac{3}{2} - 1} \right) = \left( \frac{3}{2} \right)^{m-2}, \]
and so (6) holds for \( h = m \).

On multiplying the inequalities
\[ a_1 \leq S_1 \quad \text{and} \quad a_h \leq \left( \frac{3}{2} \right)^{h-2} S_h \quad \text{for} \quad h = 2, 3, \ldots, n, \]
we have
\[ a_1 a_2 \ldots a_n \leq \left( \frac{3}{2} \right)^{1+2+\ldots+(n-2)} S_1 S_2 \ldots S_n \leq \frac{2^n \left( \frac{3}{2} \right)^{\frac{1}{2}(n-1)(n-2)}}{J}, \]
as was to be proved.

Suppose in particular that
\[ \left\{ f(x) \right\}^2 = F(x) = \sum_{h,k=1}^{n} a_{hk} x_h x_k \]
is a reduced positive definite quadratic form of determinant \( D \). Then
\[ J = \frac{\left\{ \Gamma \left( \frac{1}{2} \right) \right\}^n}{\Gamma \left( 1 + \frac{1}{2} n \right) \sqrt{D}} \]
is the volume of the convex body \( f(x) \leq 1 \), and so by our theorem
\[ a_{11} a_{22} \ldots a_{nn} \leq 2^{2n} \left( \frac{3}{2} \right)^{(n-1)(n-2)} \frac{\left\{ \Gamma \left( 1 + \frac{1}{2} n \right) \right\}^2}{\left\{ \Gamma \left( \frac{1}{2} \right) \right\}^{2n}} D, \]
since
\[ a_{hh} = a_h^2 \quad (h = 1, 2, \ldots, n). \]