The intersection formulae for a Grassmannian variety: W. V. D. Hodge.

Dirac's equation and Einstein's geometry of distant parallelism: H. W. Haskey.

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(1) Lattice points in two dimensional star domains; (2) Note on lattice points in star domains: K. Mahler.

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A table of partitions: J. A. Todd.

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NOTE ON LATTICE POINTS IN STAR DOMAINS

K. MAHLER*.

About a year ago, in a paper not yet published, Prof. Mordell proved a number of very general theorems on lattice points in finite and infinite regions bounded by concave curves. His results opened up a new domain of research, not dealt with by Minkowski's theories. They were also the more important because they could be applied to concrete cases. I refer the reader to his note, Journal London Math. Soc., 16 (1941), 149–151, for an enumeration of some of his results.

Prof. Mordell used an entirely new method, different from that which Minkowski applied to analogous questions concerning convex domains. I therefore asked myself whether Minkowski's original ideas could not be so generalized as to be applicable to non-convex domains. In a rather long paper submitted for publication in the Proceedings of the Society, I show now that this is indeed so.

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I treat the general star domain $K$, that is, a closed bounded point set of the following kind:

(a) $K$ contains the origin $O$ of the coordinate system $(x, y)$ as an inner point;

(b) the boundary $L$ of $K$ is a Jordan curve consisting of a finite number of analytical arcs;

(c) every radius vector from $O$ intersects $L$ in one, and only one, point.

I assume, further, that the domain is symmetrical about $O$, i.e. that if it contains a point $(x, y)$ it contains also the point $(-x, -y)$. The general unsymmetrical case is reduced to this symmetrical one by a trivial transformation.

A lattice $\Lambda$ of points $P$

$$(x, y) = (ah + \beta k, \gamma h + \delta k) \quad (h, k = 0, \pm 1, \pm 2, \ldots)$$

is called $K$-admissible if the origin $O$ is the only point of $\Lambda$ which is an inner point of $K$. Let

$$d(\Lambda) = |a\delta - \beta \gamma|$$

be the determinant of $\Lambda$, and $\Delta(K)$ the lower limit of $d(\Lambda)$ for all $K$-admissible lattices. It is easily proved that $\Delta(K) > 0$. I show that there always exists at least one $K$-admissible lattice $\Lambda$ such that

$$d(\Lambda) = \Delta(K),$$

a critical lattice in Prof. Mordell's notation.

I have developed, in my paper referred to above, a method by which all critical lattices of $K$ can be determined in a finite number of steps; hence $\Delta(K)$ can also be found. While this method is theoretically perfect, it may require in practice a formidable amount of work in solving systems of a finite number of equations in a finite number of unknowns.

My method, as presented, is restricted to bounded domains. I think, however, that this restriction can be removed by a simple limiting process. It seems also probable that the method can be extended to problems in three or more dimensions.

So far, I have applied the method only to a few special cases. These simple results seem to be new.
(1) The excentric ellipse. Let $K$ be an ellipse of area $J\pi$ which contains $O$ as an inner point. Let the concentric, similar, and similarly situated ellipse through $O$ be of area $J_0\pi$. Then

$$\Delta(K) = \sqrt{\frac{(J-J_0)}{2}} \{2\sqrt{(J_0)}+\sqrt{(3J+J_0)}\}.$$

I am much indebted to Mrs. W. R. Lord for solving a problem in Euclidean geometry from which I derived this value of $\Delta(K)$.

(2) The excentric parallelogram. Let $K$ be a parallelogram which contains $O$ as an inner point. Let the lines through $O$ parallel to its sides divide $K$ into four parallelograms of areas $J_1, J_2, J_3, J_4$, where the indices are chosen such that $J_1 \leq J_2 \leq J_3 \leq J_4$. Then

$$\Delta(K) = J_2 + J_3 - J_1.$$

(3) The excentric triangle. Let $K$ be a triangle which contains $O$ as an inner point. Let the lines through $O$ parallel to two of its sides, together with the third side, form triangles of areas $J_1, J_2, J_3$, where the notation is such that $J_1 \leq J_2 \leq J_3$. Then

$$\Delta(K) = 2\sqrt{(J_2 J_3)}.$$

(4) The domain $K$ obtained by combining two concentric ellipses. Let $K$ be the set of all points $(x, y)$ such that either

$$a_1 x^2 + b_1 xy + c_1 y^2 \leq 1 \quad \text{or} \quad a_2 x^2 + 2b_2 xy + c_2 y^2 \leq 1.$$  

Here the two quadratic forms on the left-hand sides are assumed to be positive definite and of determinants $1$; i.e.,

$$a_1 c_1 - b_1^2 = a_2 c_2 - b_2^2 = 1.$$  

Their simultaneous invariant is

$$J = a_1 c_2 - 2b_1 b_2 + c_1 a_2.$$  

Excluding the case when the forms are identical, we have

$$J > 2,$$

and it is easily seen that $\Delta(K) = D(J)$ is a function of $J$ only.

I develop a simple algorithm for obtaining $D(J)$ for every $J > 2$; in particular, I give the explicit value of $D(J)$ for $2 < J \leq 25$. Further, a table of the critical lattices for every $J$ in this interval is given. Both $D(J)$ and these critical lattices depend in a rather complicated way on
NOTE ON LATTICE POINTS IN STAR DOMAINS.

There are an infinity of values of $J$ for which $D(J) = \frac{1}{2} \sqrt{3}$. For all $J$,

\[ \frac{\sqrt{3}}{2} \leq D(J) \leq \frac{\sqrt{15}}{2}, \]

and

\[ \lim_{J \to \infty} D(J) = \frac{\sqrt{3}}{2}. \]

It may be remarked that $1/D(J)$ is not less than the minimum of the smaller of the two numbers

\[ a_1 x^2 + 2b_1 xy + c_1 y^2, \quad a_2 x^2 + 2b_2 xy + c_2 y^2 \]

for integral values of $x$ and $y$ not both zero.

The method used in (4) can also be applied to other domains obtained by combining two convex domains, e.g., to Prof. Mordell's star-shaped octagon (loc. cit., 149), or to that obtained from two rectangles with centres at the origin and sides parallel to the axes.

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NOTE ON THE ABSOLUTE SUMMABILITY OF TRIGONOMETRICAL SERIES

Fu Traing Wang*.

A series $\sum A_n$ is said† to be summable $|A|$ if $F(r) = \sum A_n r^n$ is of bounded variation in the interval $0 < r < 1$. A series which is summable $|C|$ is also‡ summable $|A|$, but one which is summable $|A|$ need not be summable (C), as is shown by the well-known example $F(r) = e^{(1+r)^{1/2}}$, while a convergent series need not§ be summable $|A|$.

Necessary and sufficient conditions for the summability $|C|$ of a Fourier series have been given by Bosanquet||. On the other hand, the author has proved the following result¶.

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§ Whittaker, loc. cit.