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fying the canonical equation (11) identically, on substituting in (15), (16), (22) for $\xi, \eta, \zeta$ from (26) and (25). The unicursal curve represented by this parametric solution of (11) is clearly of order 18. From a previous general result*, it is the complete intersection of the cubic surface (11) and another algebraic surface of order six.

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ON LATTICE POINTS IN AN INFINITE STAR DOMAIN

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In my paper "On lattice points in star domains", which is to appear in the Proceedings of the London Mathematical Society, I defined a finite star domain by the following properties:

1. The domain $K$ is a bounded closed point set in the $(x, y)$-plane.
2. $K$ contains the origin $O = (0, 0)$ as an inner point.
3. The boundary $C$ of $K$ is a Jordan curve.
4. Every radius vector from $O$ intersects $C$ in just one point.
5. If $K$ contains the point $P = (x, y)$, then it also contains the point $-P = (-x, -y)$ symmetrical to $P$ in $O$.

I called a lattice

$(\Lambda) (x, y) = (ah + \beta k, \gamma h + \delta k) \ (a, \beta, \gamma, \delta$ real numbers; $h, k = 0, \pm 1, \ldots)$

$K$-admissible, if $O$ is the only inner point of $K$ belonging to $\Lambda$. Then

$d(\Lambda) = |a\delta - \beta\gamma|$

is called the determinant of $\Lambda$, and $\Delta(K)$ denotes the lower bound of $d(\Lambda)$ for all $K$-admissible lattices. It was shown that $\Delta(K) > 0$, and that there exists at least one critical lattice, i.e. a $K$-admissible lattice $\Lambda$ such that $d(\Lambda) = \Delta(K)$. It was further proved trivially that if the finite star

* Cf. B. Segre, "A note on arithmetical properties of cubic surfaces", loc. cit., Theorem VII.
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domain $K$ is contained in the finite star domain $K'$, then

$$\Delta(K) \leq \Delta(K').$$

In this note, I consider infinite star domains, i.e. point sets $K$ in the $(x, y)$-plane such that

"If $K_r$ is, for every positive number $r$, the set of all those points of $K$ which have a distance not greater than $r$ from $0$, then $K_r$ is a finite star domain".

If $r < r'$, then $K_r$ is contained in $K_{r'}$; hence

$$\Delta(K_r) \leq \Delta(K_{r'}).$$

Therefore $\Delta(K_r)$ is an increasing function of $r$. Put

$$\Delta(K) = \lim_{r \to \infty} \Delta(K_r).$$

If $\Delta(K) = \infty$, then every lattice contains an infinity of points of $K$; an example of a domain of this kind is given by

$$|xy| \leq \delta/5, \ |x| < \epsilon$$

is solvable for every $\epsilon > 0$. In this note, I assume from now on that $\Delta(K)$ is finite.

**Theorem 1.** There exists at least one critical lattice $\Lambda$ of $K$, i.e. a lattice with the following properties:

1. $O$ is the only inner point of $K$ belonging to $\Lambda$.
2. $d(\Lambda) = \Delta(K)$, i.e. $= \lim_{r \to \infty} \Delta(K_r)$. [This differs from the definition of $\Delta(K)$ for finite domains.]
3. There is no $K$-admissible lattice of determinant less than $\Delta(K)$. [I.e. the definition of $\Delta(K)$ in (2) is equivalent to that in the case of a finite domain.]

**Proof.** The origin is an inner point of $K$; there is therefore a positive number $\rho$ such that the circle $K$ of centre $O$ and radius $\rho$ lies entirely in $K_1$. Hence $K$ is also a subset of $K_n$ for $n = 1, 2, 3, \ldots$.

Denote by $\Lambda_n$ a critical lattice of $K_n$, and by $R_n, S_n$ a basis of $\Lambda_n$. This basis can be chosen so as to be reduced; i.e. all angles of the parallelogram with vertices at $O, R_n, R_n + S_n, S_n$ lie between $60^\circ$ and $120^\circ$. Then
by a well-known property of reduced lattices,
\[ \sqrt{\frac{2}{3}} \overline{OR_n} \overline{OS_n} \leq d(\Lambda_n) = \Delta(K_n) \leq \Delta(K). \]
Further, since no element of \( \Lambda_n \) can be an inner point of \( K \),
\[ \overline{OR_n} \geq \rho, \quad \overline{OS_n} \geq \rho. \]
Hence
\[ \overline{OR_n} \leq \frac{2\Delta(K)}{\rho \sqrt{3}}, \quad \overline{OS_n} \leq \frac{2\Delta(K)}{\rho \sqrt{3}}, \]
and so the two basis points \( R_n, S_n \) of \( \Lambda_n \) lie at a bounded distance from \( O \).
Hence there exists an infinite sequence of indices
\( n_1, n_2, n_3, \ldots \),
such that the two basis points
\[ R_n, S_n \quad (n = n_1, n_2, n_3, \ldots) \]
tend to limit points \( R \) and \( S \), respectively.
Denote by \( \Lambda \) the lattice of basis \( R, S \). Then
\[ d(\Lambda) = \lim_{r \to \infty} d(\Lambda_n) = \Delta(K). \]
This lattice \( \Lambda \) is \( K \)-admissible. For if this be false, let
\[ P = hR + kS \quad (h, k \text{ integers}) \]
be a point of \( \Lambda \) different from \( O \) which is an inner point of \( K \). The sequence of points
\[ P_n = hR_n + kS_n \quad (n = n_1, n_2, n_3, \ldots) \]
tends to \( P \), and so \( P_n \) is arbitrarily near to \( P \) for large \( n \). Hence also \( P_{n_v} \) is an inner point of \( K \) if \( v \) is sufficiently large. Let \( r \) be the distance of \( P \) from \( O \). Then, for \( n_v > r \), \( P \) is also an inner point of \( K_{n_v} \). This, however, is contrary to the assumption that \( \Lambda_{n_v} \) is a \( K_{n_v} \)-admissible lattice.
There cannot be a \( K \)-admissible lattice \( \Lambda^* \) for which
\[ d(\Lambda^*) < \Delta(K). \]
For, if such a lattice should exist, let \( n \) be an index such that
\[ d(\Lambda^*) < \Delta(K_n). \]
Then at least one point \( P \neq O \) of \( \Lambda^* \) is an inner point of \( K_n \), and hence an inner point of \( K \), contrary to hypothesis. This completes the proof.
It was proved in my paper that every critical lattice of a finite star domain has at least four points on its boundary. This is not so for infinite domains.

**Theorem 2.** There exists an infinite star domain $K$ of boundary $C$ such that no critical lattice of $K$ has a point on $C$.

**Proof.** Denote by $K$ a domain with the properties:

1. $K$ is an infinite star domain.
2. All points of $K$ are inner points of the infinite star domain $K^*$ defined by
   \[ |xy| \leq 1. \]
3. If the point $P = (x, y)$ on $C$ is at the distance $r$ from $O$, then
   \[ \lim_{r \to \infty} |xy| = 1. \]

By a theorem of Hurwitz,

\[ \Delta(K^*) = \sqrt{5}; \]

hence, since $K$ is contained in $K^*$,

(I) \[ \Delta(K) \leq \Delta(K^*) = \sqrt{5}. \]

Let further $\epsilon$ and $t$ be two positive numbers, of which $\epsilon$ is sufficiently small, and denote by $K(\epsilon, t)$ the finite star domain

\[ |xy| \leq (1 - \epsilon)^2, \quad |tx + \frac{1}{t} y| \leq \sqrt{5}(1 - \epsilon). \]

Then, by a theorem of mine (in my paper: “On lattice points in the star domain $|xy| \leq 1, |x+y| \leq \sqrt{5}$”, which is to appear in the Proceedings of the Cambridge Philosophical Society),

\[ \Delta(K(\epsilon, t)) = \sqrt{5}(1 - \epsilon)^2 \]

is independent of the value of $t$. I assert that, for all sufficiently large $t$, $K(\epsilon, t)$ is contained in $K$, so that

(II) \[ \Delta(K) \geq \Delta(K(\epsilon, t)) = \sqrt{5}(1 - \epsilon)^2. \]
For choose a positive number $r(\epsilon)$ such that

$$|xy| > (1 - \epsilon)^2$$

for all points $P$ of $C$ for which $r > r(\epsilon)$; such a constant exists by the property (3) of $K$. It is clear from this definition of $r(\epsilon)$ that no point $P$ on $C$ with $r > r(\epsilon)$ belongs to $K(\epsilon, t)$; hence it suffices to show that no point on $C$ with $r \leq r(\epsilon)$ belongs to $K(\epsilon, t)$. Now the two coordinate axes are asymptotes of $C$, but do not intersect $C$. Hence there exists a positive number $\delta(\epsilon)$ such that

$$|x| \geq \delta(\epsilon), \quad |y| \leq r$$

for all points $P = (x, y)$ on $C$ with $r \leq r(\epsilon)$. Choose $t$ so large that

$$t > \frac{1 + \sqrt{5}}{\delta(\epsilon)}, \quad t > r;$$

then

$$\left| tx + \frac{1}{t} y \right| > \frac{1 + \sqrt{5}}{\delta(\epsilon)} \delta(\epsilon) - \frac{1}{r} r = \sqrt{5} > \sqrt{5(1 - \epsilon)},$$

as asserted.

Since $\epsilon$ may be arbitrarily small, from (I) and (II),

$$\Delta(K) = \sqrt{5}.$$ 

Hence, if

$$(\Lambda) \quad (x, y) = (\alpha h + \beta k, y_\delta + \delta k) \quad (h, k = 0, \pm 1, \pm 2, \ldots)$$

is a critical lattice of $K$, then

$$d(\Lambda) = |\alpha \delta - \beta y| = \sqrt{5}.$$ 

To $\Lambda$, we make correspond the indefinite quadratic form

$$\Phi(h, k) = (\alpha h + \beta k)(\gamma h + \delta k) = \alpha h^2 + 2bhk + c(k^2$$

of determinant

$$b^2 - ac = \left(\frac{\alpha \delta - \beta y}{2}\right)^2 = \frac{5}{4}.$$ 

By the property (3) of $K$, this form satisfies the inequality

$$|\Phi(h, k)| \geq (1 - \epsilon)^2$$

for all integers $h, k$ with sufficiently large $h^2 + k^2$. 
Now, by a theorem of Markoff, the forms equivalent to
\[ \pm (h^2 - hk - k^2) \]
are the only ones of determinant \( \frac{5}{3} \) which do not assume values numerically less than 1 for integral \( h, k \) not both zero; every other form of determinant \( \frac{5}{3} \) represents numbers numerically not greater than
\[ \sqrt{\frac{5}{3}} \]
for an infinity of integral \( h, k \).

Hence, since \( \epsilon \) may be chosen so small that
\[ (1 - \epsilon)^2 > \sqrt{\frac{5}{3}}, \]
\( \Phi(h, k) \) must be the form
\[ \Phi(h, k) = \pm (h^2 - hk - k^2), \]
and so
\[ |xy| = |\Phi(h, k)| \geq 1 \]
for all points of \( \Lambda \) different from 0. Therefore, as asserted, no point of \( \Lambda \) lies on the boundary \( C \) of \( K \). We also see that \( K \) has actually an infinity of critical lattices, namely
\[ (x, y) = \left( \lambda \left( h - \frac{1+\sqrt{5}}{2} k \right), \frac{1}{\lambda} \left( h - \frac{1-\sqrt{5}}{2} k \right) \right) \quad (h, k = 0, \pm 1, \pm 2, \ldots), \]
where \( \lambda \) is any positive or negative number.

A slight modification of this proof proves that suitable infinite star domains possess critical lattices with any even number of points on \( C \).

There is no difficulty in extending Theorem 1 to more than two dimensions, if use is made of the theory of reduced quadratic forms to find \( n \) points forming a basis of the lattice.

Theorem 2 has an analogue in three dimensions, as can be deduced from results of Davenport on the product of three linear forms, but I do not know whether such an analogue holds in more than three dimensions†.

January, 1944. I have recently extended the result of this note to more dimensions, and proved some further existence theorems.

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