assume that $R$ is different from the ten points $\pm P_1, \pm P_2, \pm (2P_2 - P_1), \pm (P_2 - P_1), \pm (P_2 - 2P_1)$ of $A_0$ on $L$; for otherwise we may replace $R$ by a neighbouring point on $L$ having this property and lying outside $H$.

By Theorem 2, there exists a critical lattice $\Lambda$ of $K$ which contains the point $R$ on $L$. This lattice is also $H$-admissible. It is, however, not a critical lattice of $H$. For $\Lambda$ contains six points on the boundary $L$ of $K$, and so at most four points on the boundary of $H$; and so $\Lambda$ would be a singular lattice of $H$. Then the tac-line conditions (see the preface) must be satisfied by the four points on the boundary of $H$. These four points lie also on the boundary of $K$, and we have shown in the proof of Theorem 1 that the tac-line conditions never hold for points on the boundary of $K$. Hence $\Lambda$ is not a critical lattice of $H$, and so there exist critical lattices of smaller determinant than $\Lambda(K)$, as was to be proved.

I wish to express my thanks to Prof. Mordell and Prof. Hardy for their help with the manuscript.

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ON LATTICE POINTS IN THE DOMAIN $|xy|<1, |x+y|<\sqrt{5}$, AND APPLICATIONS TO ASYMPTOTIC FORMULAE IN LATTICE POINT THEORY (II)

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Communicated by L. J. MORDELL
Received 25 June 1943

I. LATTICE POINTS IN THE DOMAIN $|x|^\alpha + |y|^\alpha \leq 1$

THEOREM 4. Let $G$ be the star domain $|x|^\alpha + |y|^\alpha \leq 1$
where $\alpha > 0$. Then, when $\alpha$ tends to zero,
$$\Delta(G) = 2^{-2\alpha} \sqrt{5} \{1 + O(\alpha)\}.$$

Proof. The linear substitution $x = 2^{-1/\alpha} X, y = 2^{-1/\alpha} Y$
changes $G$ into the similar domain
$$(G')$$
$|X|^\alpha + |Y|^\alpha \leq 2$,
and so
$$\Delta(G) = 2^{-2\alpha} \Delta(G').$$

Now $|X|^\alpha + |Y|^\alpha = e^{\alpha \log|X| + \epsilon^\alpha \log|Y|} = 2 + \alpha \log|XY| + \rho(X,Y)$,
where, by the mean value theorem of the differential calculus,
$$\rho(X,Y) = \frac{1}{2} \alpha^2 \{e^{\alpha \log|X|} (\log|X|)^2 + e^{\alpha \log|Y|} (\log|Y|)^2\}$$
with $0 < \theta < 1$. Hence, for all points on the boundary of $G'$,
$$\log|XY| = -\frac{\rho(X,Y)}{\alpha}, \quad \text{i.e.} \quad |XY| = e^{-\rho(X,Y)/\alpha} \leq 1.$$
This means that $G'$ is a subdomain of the domain $K'$ defined by $|XY| \leq 1$; and so, by Hurwitz's theorem,

$$(A) \quad \Delta(G') \leq \Delta(K') = \sqrt{5}.$$  

On the other hand, if $\alpha > 0$ is sufficiently small, and $c > 0$ is a suitable constant, then the domain

$$(K_\alpha) \quad |XY| \leq (1-c\alpha)^2, \quad |X+Y| \leq \sqrt{5}(1-c\alpha),$$

is contained in $G'$. For, if $(X, Y)$ is a point in $K_\alpha$, then

$$|X-Y| = +\sqrt{(X+Y)^2 - 4XY} < +\sqrt{(5+4)} = 3,$$

and so $\max(|X|, |Y|) = \frac{1}{2}(|X+Y| + |X-Y|) < \frac{1}{2}(3 + \sqrt{5}) < e.$

If now $|X| \leq e^{-2}$ or $|Y| \leq e^{-2}$, then for sufficiently small positive $\alpha$,

$$|X|^a + |Y|^a < e^a + e^{-2a} = 1 - \alpha + O(\alpha^2) < 1,$$

and so $(X, Y)$ is an inner point of $G'$. We may therefore assume that the point $(X, Y)$ in $K_\alpha$ satisfies the inequalities

$$e^{-2} \leq |X| \leq e, \quad e^{-2} \leq |Y| \leq e.$$

But then

$$0 \leq \rho(X, Y)/\alpha \leq \frac{1}{2}\alpha(\alpha^2 - 2\alpha^2) = 4\alpha e^a,$$

and so, if we choose $c = 5, \quad e^{-\rho(X,Y)/\alpha} > e^{-\alpha} > (1 - \alpha)^2,$

if $\alpha > 0$ is again sufficiently small. The assertion follows therefore also in this case.

By Theorem 1 of part I,

$$\Delta(K_\alpha) = (1-c\alpha)^2 \Delta(K) = (1-c\alpha)^2 \sqrt{5},$$

since $K_\alpha$ is derivable from $K$ by the linear substitution

$$X = x'(1-c\alpha), \quad Y = y'(1-c\alpha)$$

of determinant $(1-c\alpha)^2$. Hence

$$(B) \quad \Delta(G') \geq \Delta(K_\alpha) = (1-c\alpha)^2 \sqrt{5}.$$  

Theorem 4 is now an immediate consequence of $(A)$ and $(B)$.

II. ON POSITIVE DEFINITE QUARTIC BINARY FORMS

1. The problem. Let

$$f(x, y) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4$$

be a positive definite quartic binary form of invariants

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2; \quad g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}; \quad G = g_2^3 - 27g_3^2; \quad J = \frac{g_3^3}{G} = 27 \frac{g_3^3}{G} + 1;$$

here $G$ is the discriminant, and $J$ the absolute invariant of $f(x, y)$. Denote by $K_f$ the star domain

$$f(x, y) \leq 1,$$

and by $\Delta(K_f)$ the lower bound of the determinants of all $K_f$-admissible lattices. Then $\Delta(K_f)$ is an invariant of $f(x, y)$, and so is a function of $g_2$ and $g_3$ alone.

Since $f(x, y)$ is positive definite, its discriminant is positive. We shall therefore assume from now on that $G = 1$. Then $g_2$ and $g_3$ become functions of $J$ and of the sign

$$\epsilon = \text{sgn} g_3 = +1(g_3 > 0), \quad 0(g_3 = 0), \quad -1(g_3 < 0),$$

namely

$$g_2 = J^4, \quad g_3 = \epsilon \left( \frac{J - 1}{27} \right)^{\frac{1}{4}},$$
with the convention that all square and higher roots are taken with the positive sign. Hence also
\[ \Delta(K_f) = D_e(J), \]
say, depends only on the values of \( J \) and \( \epsilon \). Our problem is to obtain an asymptotic formula for \( D_e(J) \) when \( J \) tends to infinity, and to study the minimum of \( f(x, y) \) for integral \( x, y \), not both zero, when \( J \) is large.

2. The normal form of \( f(x, y) \). Since \( f(x, y) \) is positive definite, it is the product of two positive definite binary quadratic forms. By means of a linear substitution of unit determinant, these factors can be reduced to the forms
\[ x^2 + y^2, \quad ax^2 + by^2 \quad (a > 0, \ b > 0). \]

Hence we may assume that
\[ f(x, y) = (x^2 + y^2)(ax^2 + by^2), \]
where \( a > 0, \ b > 0 \) satisfy the condition that \( f(x, y) \) is of discriminant
\[ G = ab(a - b)^4 = 1. \]

This condition \( G = 1 \) is equivalent to
\[ a = 2A\{4AB(A - B)^4\}^{-1}, \quad b = 2B\{4AB(A - B)^4\}^{-1}, \]
where \( A \) and \( B \) are any two positive numbers.

We therefore suppose from now on that
\[ f(x, y) = 2\{4AB(A - B)^4\}^{-1}(x^2 + y^2)(Ax^2 + By^2) \quad (A > 0, \ B > 0). \]

Then the invariants of \( f(x, y) \) are
\[ g_2 = \frac{A^2 + 14AB + B^2}{3\{4AB(A - B)^4\}^{1/4}}, \quad g_3 = \frac{- (A + B)(A^2 - 34AB + B^2)}{27\{4AB(A - B)^4\}^{1/4}}, \]
\[ J = g_3^2 = 27g_3^2 + 1. \]

Hence \( J \) tends to infinity only if \( A/B \) tends to either 0 or 1 or \( \infty \).

3. The case \( A/B \to 1 \). Put \( B = A(1 + \epsilon) \),
where \( \epsilon \to 0 \). Then
\[ g_2 = \frac{16}{3\cdot 4^4} t^{-4}\{1 + O(\epsilon t)\}, \quad g_3 = \frac{32}{27\cdot 4^4} t^{-4}\{1 + O(\epsilon t)\}, \quad J = 2(\frac{\epsilon}{4})^3 t^{-4}\{1 + O(\epsilon t)\}, \]
and, conversely,
\[ t = \pm 2t^{\frac{2}{6}} J^{-1}\{1 + O(J^{-1})\}. \]

It is clear that \( g_3 > 0 \) for all sufficiently small values of \( \epsilon t \), i.e. for all sufficiently large values of \( J \); and therefore \( \epsilon = \text{sgn} g_3 = +1 \).

The quartic \( f(x, y) \) takes now the form
\[ f(x, y) = \frac{2(x^2 + y^2)(x^2 + (1 + \epsilon t)y^2)}{\{4t^4(1 + \epsilon t)\}^{1/4}}, \]
whence
\[ \frac{2(1 + \epsilon t)}{\{4t^4(1 + \epsilon t)\}^{1/4}} (x^2 + y^2)^2 \leq f(x, y) \leq \frac{2(1 + (1 + \epsilon t)^2)}{\{4t^4(1 - \epsilon t)\}^{1/4}} (x^2 + y^2)^2. \]
for all real values of \( x \) and \( y \). Hence \( K_f \) is contained in the circle
\[ (C_4) \quad x^2 + y^2 \leq \frac{\{4t^4(1 + \epsilon t)\}^{1/4}}{\{2(1 - \epsilon t)\}^{1/4}} = 2^{-1/4} |t|^{1/4}\{1 + O(\epsilon t)\} = (\frac{\epsilon}{4})^{1/4} J^{-1/4}(1 + O(J^{-1})). \]
Lattice points

and itself contains the circle

\[(C_2) \quad x^2 + y^2 \leq \frac{4t^4(1 - |t|)}{(2(1 + |t|))^{3/4}} = 2^{-1} |t|^{3/4} \{1 + O(||t||)\} = \left(\frac{4}{3}\right)^{1/2} J^{-1/12} \{1 + O(J^{-1})\}.
\]

Now it is well known that, for a circle \(x^2 + y^2 \leq r^2\), the minimum determinant of all admissible lattices is given by

\[\Delta(C) = \left(\frac{4}{3}\right)^{1/2} r^2.
\]

Hence

\[\sqrt{\left(\frac{3}{4}\right) \left\{ \frac{4t^4(1 - |t|)}{(2(1 + |t|))^{3/4}} \right\}^{1/12}} \leq D_4(J) \leq \sqrt{\left(\frac{3}{4}\right) \left\{ \frac{4t^4(1 + |t|)}{(2(1 - |t|))^{3/4}} \right\}^{1/12}},
\]

and so finally

(I) \[D_{-1}(J) = \left(\frac{4}{3}\right)^{1/2} J^{-1/12} \{1 + O(J^{-1})\} \]
as \(J \to \infty\).

4. The case \(A/B \to 0 \text{ or } \infty\). Since \(f(x, y)\) is symmetrical in \(A\) and \(B\), it suffices to consider the case when \(A/B\) tends to infinity. Put therefore

\[B = At,
\]

where \(t > 0\) and \(t \to 0\). Then

\[g_2 = \frac{1 + O(t)}{3(4t)^{3/4}}, \quad g_3 = -\frac{1 + O(t)}{27(4t)^{3}}, \quad J = \frac{1 + O(t)}{108t},
\]

and conversely,

\[t = \frac{1 + O(J^{-1})}{108J}.
\]

It is clear that \(g_3 < 0\) for all sufficiently small values of \(t\), i.e. for all sufficiently large values of \(J\); and therefore \(c = \text{sgn} \, g_3 = -1\).

Now \(f(x, y)\) takes the form

\[f(x, y) = \frac{2(x^2 + y^2)(x^2 + ty^2)}{4t(1-t)^{3/2}}.
\]

Put

\[x = t^{1/4}X, \quad y = t^{-1/12}Y;
\]

these formulae define a linear substitution of determinant \(t^{1/12}\). By this substitution, \(f(x, y)\) is changed into the new form

\[f(x, y) = F(X, Y) = \left(\frac{16}{(1-t)^4}\right)^{1/4} \{1 + t\} X^2 Y^2 + \sqrt{t} (X^4 + Y^4),
\]

and \(K_f\) is changed into the new star domain

\[F(X, Y) \leq 1, \quad i.e. \quad X^2 Y^2 \leq \frac{1}{1+t}\left\{ (1-t)^{4/4} - \sqrt{t} (X^4 + Y^4) \right\}.
\]

By the relation connecting \(K_f\) with \(K'_f\),

\[D_{-1}(J) = \Delta(K_f) = t^{1/12} \Delta(K'_f).
\]

Now \(K'_f\) is contained in the star domain

\[|XY| \leq 2^{-4}.
\]

On the other hand, if \(t > 0\) is sufficiently small, \(c > 0\) denotes a suitable absolute constant, and \(X, Y\) are bounded independently of \(t\) and \(c\), then

\[\frac{1}{1+t}\left\{ (1-t)^{4/4} - \sqrt{t} (X^4 + Y^4) \right\} \geq 4^{-1} (1-c \sqrt{t})^2
\]
in \(K'_f\). Hence \(K'_f\) contains the star domain
for which obviously $X, Y$ are bounded independently of $t$ and $c$.

Further, by Hurwitz's theorem,
\[ \Delta(H_1) = 2^{-\frac{1}{3}}, \]
and by Theorem 1 of Part I, \( \Delta(H_2) = 2^{-\frac{1}{3}} \sqrt{5} \).

Hence, finally,
\[ 2^{-\frac{1}{3}} \sqrt{5} \leq D_{-1}(J) \leq 2^{-\frac{1}{3}} \sqrt{5} \mu_{12}, \]
and, on replacing $t$ by its value as a function of $J$,
\[ D_{-1}(J) = (\frac{\mu_{12}}{\sqrt{5}})^4 J^{-1/12} \{1 + O(J^{-1})\} \]
as $J \to \infty$.

5. The minimum of $f(x, y)$. By definition, $D_\epsilon(J)$ is the lower bound of the determinants of the $K_\epsilon$-admissible lattices. Consider now the similar domain
\[ (K_{\epsilon,s}) \]
where $s > 0$. Since it can be obtained from $K_\epsilon$ by the linear substitution
\[ \bar{x} = s^4 x, \quad \bar{y} = s^4 y, \]
we find the equation
\[ \Delta(K_{\epsilon,s}) = s^4 D_\epsilon(J). \]
Hence, if in particular $s = D_\epsilon(J)^{-2}$, then $\Delta(K_{\epsilon,s}) = 1$, i.e. every lattice of determinant 1 contains at least one point, other than the origin, of the domain
\[ f(x, y) \leq D_\epsilon(J)^{-2}. \]
Further, there exist lattices of determinant 1 which contain no inner points, other than the origin, of this domain.

All lattices of determinant 1 can be obtained from the lattice of all points with integral coordinates by linear substitutions of unit determinant. Hence, by the invariance of $J$ and $\epsilon$, and by the formulae (I) and (II), we arrive at the following result.

**Theorem 5.** Let $\Sigma_\epsilon(J)$ be the set of all positive definite binary quartic forms $f(x, y)$ of discriminant $G = 1$, absolute invariant $J$, and of invariant $g_3$ satisfying
\[ \text{sgn} \, g_3 = \epsilon. \]
Let further $m(f)$ be the minimum of $f(x, y)$ for all pairs of integers $x, y$ not both zero, and let $M_\epsilon(J)$ be the upper bound of $m(f)$ extended over all forms in $\Sigma_\epsilon(J)$. Then, when $J$ tends to infinity,
\[ M_\epsilon(J) = \begin{cases} \sqrt{(\frac{1}{3}) J^4 (1 + O(J^{-1}))} & \text{if } \epsilon = +1, \\ \sqrt{(\frac{1}{3}) J^4 (1 + O(J^{-1}))} & \text{if } \epsilon = -1. \end{cases} \]
In both cases there exist forms $f(x, y)$ such that $m(f) = M_\epsilon(J)$.

This theorem shows that $M_\epsilon(J)$ tends to infinity with $J$. In the other direction, it is not difficult to show that $M_\epsilon(J)$ has a positive lower bound, namely,
\[ M_\epsilon(J) \geq \frac{1}{3} (432)^4. \]
But this is presumably not the exact lower bound, and there is a possibility that this lower bound is attained if $J = 1$. 

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