LATTICE POINTS IN TWO-DIMENSIONAL STAR DOMAINS (III)

By Kurt Mahler.

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If

\[ f(x, y) = ax^2 + 2bxy + cy^2 \]

is a positive definite binary quadratic form of determinant

\[ ac - b^2 = 1, \]

and \( E \) denotes the domain

\[ f(x, y) \leq 1, \]

bounded by the ellipse \( f(x, y) = 1 \), then by a classical result\(^\dagger\),

\[ \Delta(E) = \sqrt{\frac{8}{2}}. \]

There exists a continuous infinity of critical lattices \( \Lambda \). Every such lattice contains just six points \( \pm P_1, \pm P_2, \pm P_3 \) on the boundary of \( E \). It is possible to choose the notation such that

\[ P_1 + P_2 + P_3 = 0. \]

Conversely, six arbitrary boundary points of this type generate a critical lattice, any two independent points among them forming a basis.

The present fourth chapter of this paper deals with the more complicated domain \( K \) obtained by combining two concentric ellipses each of area \( \pi \). An algorithm is developed for determining \( \Delta(K) \), which turns out to be a rather complicated function of the simultaneous invariant of the two ellipses.

A similar method can be applied to all domains obtained by combining two convex domains with centre at \( O \), e.g. the star-shaped octagon investigated by Prof. Mordell.

\(^\dagger\) Bachmann, *Quadratische Formen*, II (Leipzig und Berlin, 1923), Kap. 5.
CHAPTER IV. THE DOMAIN BOUNDED BY TWO ELLIPSES.

25. The invariant $J$.

Let

\[(50)\quad f_1(x, y) = a_1x^2 + 2b_1xy + c_1y^2 \quad \text{and} \quad f_2(x, y) = a_2x^2 + 2b_2xy + c_2y^2\]

be two positive definite binary quadratic forms of determinants

\[(51)\quad a_1c_1 - b_1^2 = a_2c_2 - b_2^2 = 1.\]

Further, let

\[(52)\quad J = a_1c_2 - 2b_1b_2 + c_1a_2\]

be the simultaneous invariant of these two forms. If an affine transformation of determinant unity,

\[(53)\quad x = ax' + \beta y', \quad y = \gamma x' + \delta y', \quad \text{where} \quad a\delta - \beta\gamma = 1,\]

changes $f_1$ and $f_2$ into the new forms

\[f'_1(x', y') = a'_1x'^2 + 2b'_1x'y' + c'_1y'^2\]

and

\[f'_2(x', y') = a'_2x'^2 + 2b'_2x'y' + c'_2y'^2,\]

then by the invariantive property of the determinants and of $J$,

\[a'_1c'_1 - b'_1^2 = a'_2c'_2 - b'_2^2 = 1, \quad a'_1c'_2 - 2b'_1b'_2 + c'_1a'_2 = J.\]

It is always possible to choose the transformation (53) so that $f'_1$ and $f'_2$ take the canonical forms

\[(54)\quad f'_1(x', y') = x'^2 + y'^2 \quad \text{and} \quad f'_2(x', y') = \lambda x'^2 + \frac{1}{\lambda} y'^2,\]

where $\lambda$ is a positive number. In this case

\[(55)\quad J = \lambda + \frac{1}{\lambda}.\]

I assume in this chapter that $f_1$ and $f_2$, and so also $f'_1$ and $f'_2$, are not identical. Hence $\lambda \neq 1$, and therefore, from (55),

\[(56)\quad J > 2.\]

We may further suppose without loss of generality that $\lambda > 1$. 
26. *The domain K.*

Let now $K$ be the domain of all points $(x, y)$ satisfying at least one of the two inequalities

$$f_1(x, y) \leq 1 \quad \text{and} \quad f_2(x, y) \leq 1.$$  

Hence $K$ is formed by combining two concentric ellipses each of area $\pi$. It is evident that $K$ is a simple star domain; we can then consider the lower bound $\Delta(K)$.

The affine transformation (53) changes $K$ into a domain $K'$ formed by the points $(x', y')$ satisfying at least one of the inequalities

$$f_1'(x', y') \leq 1 \quad \text{and} \quad f_2'(x', y') \leq 1.$$  

Hence $K'$ is of the same type as $K$.

We can assert that

$$\Delta(K) = \Delta(K').$$

For (53) changes $K$-admissible lattices into $K'$-admissible lattices, and critical lattices of $K$ into critical lattices of $K'$; and it leaves the determinant of two points and so also the determinant of a lattice invariant.

Choose the transformation (53) so that $f_1, f_2$ change into the two forms (54). Then $K'$ becomes the set of all points $(x', y')$ for which at least one of the inequalities

$$x'^2 + y'^2 \leq 1 \quad \text{and} \quad \lambda x'^2 + \frac{1}{\lambda} y'^2 \leq 1$$

holds. Here $\lambda$ is determined uniquely as a function of $J$ by

$$\lambda = \frac{J + \sqrt{(J^2 - 4)}}{2}.$$  

Hence the lower bound $\Delta(K) = \Delta(K')$ becomes a function of $J$, say

$$\Delta(K) = D(J).$$

27. *A property of the critical lattices.*

By the last paragraph, we may assume from now on that

$$f_1(x, y) = x^2 + y^2, \quad f_2(x, y) = \lambda x^2 + \frac{1}{\lambda} y^2.$$  

The two ellipses $f_1 = 1$ and $f_2 = 1$ intersect at the four points

$$Q_1: (\mu, \nu), \quad Q_2: (-\mu, \nu), \quad Q_3: (-\mu, -\nu), \quad Q_4: (\mu, -\nu),$$

where

$$\mu = \sqrt{\left(\frac{1}{\lambda+1}\right)}, \quad \nu = \sqrt{\left(\frac{\lambda}{\lambda+1}\right)}.$$
Denote by $C_1$ and $C_2$ those arcs of $f_1 = 1$ and $f_2 = 1$, respectively, which together form the boundary $C = C_1 + C_2$ of $K$. Hence, on describing $C$ in a positive direction, the arc of $C$

- from $Q_4$ to $Q_1$ belongs to $C_1$,
- from $Q_1$ to $Q_2$ belongs to $C_2$,
- from $Q_2$ to $Q_3$ belongs to $C_1$,
- from $Q_3$ to $Q_4$ belongs to $C_2$.

We use the convention of counting every one of the four points $Q_1, Q_2, Q_3, Q_4$ twice, once in $C_1$ and once in $C_2$.

The affine transformation of determinant unity,

$$x \to \lambda^{-1} y, \quad y \to \lambda^t x,$$

evidently transforms $K$ into itself, interchanges the parts $C_1$ and $C_2$ of $C$, and permutes the points $Q_1, Q_2, Q_3, Q_4$ cyclically, and by the last paragraph it changes critical lattices again into critical lattices. Hence to every critical lattice with just $m$ points on $C_1$ and $n$ points on $C_2$ there corresponds a second critical lattice with just $n$ points on $C_1$ and $m$ points on $C_2$.

**Theorem 23.** A critical lattice $\Lambda$ of $K$ has at most six points on $C_1$. If it contains six points on $C_1$, then these are of the form $\pm P_1, \pm P_2, \pm P_3$, where $P_1 + P_2 + P_3 = 0$. Further,

$$\Delta(\Lambda) = d(\Lambda) = \sqrt{2},$$

and there are also six lattice points of the same type on $C_2$.

**Proof.** The lattice $\Lambda$ is admissible with respect to the circle $f_1 \leq 1$, and so, by the introduction, cannot contain more than six points on its boundary. If it has six points on $C_1$, then these are of the mentioned form, and the lattice is critical with respect to the circle; hence (60) is satisfied. Then $\Lambda$ must also be critical with respect to the ellipse $f_2 \leq 1$; for otherwise, since $d(\Lambda) = \sqrt{2}$, at least one lattice point $P \neq 0$ would be an inner point of the ellipse and so also an inner point of $K$. Hence there are also exactly six points of $\Lambda$ on $C_2$.

**Theorem 24.** Let $\Lambda$ be a critical lattice with less than six points on $C_1$. Then there are just four lattice points $\pm P_1, \pm P_2$ on $C_1$, and four lattice points $\pm P_3, \pm P_4$ on $C_2$.

† It is possible for some of the lattice points on $C_1$ to be identical with lattice points on $C_2$. This happens when some of the points $Q_1, Q_2, Q_3, Q_4$ are lattice points.
Proof. First, let $\Lambda$ be a singular lattice. Then, by Theorem 14, its only points on $C$ are $Q_1$, $Q_2$, $Q_3$, $Q_4$; the assertion is therefore true. Secondly, let $\Lambda$ be regular; then it has at least six points on $C$. We may assume, by the last theorem, that there are just four points of $\Lambda$ on $C_1$; otherwise we apply the transformation (59) and thus obtain a regular lattice with this property.

Let, then, the four lattice points on $C_1$ be $\pm P_1$, $\pm P_2$, and assume that there are only two symmetrical lattice points $\pm P_3$ on $C_2$. Then at most one of the two pairs of symmetrical points $Q_1, Q_3$ and $Q_2, Q_4$ belong to $\Lambda$. Hence there exists a sufficiently small angle $\alpha$ such that the rotation

$$x \rightarrow x \cos \alpha - y \sin \alpha, \quad y \rightarrow x \sin \alpha + y \cos \alpha$$

changes $\Lambda$ into a new lattice $\Lambda^\times$ with only four points $\pm P_1^\times$, $\pm P_2^\times$ on $C_1$ and containing no further points $P \neq O$ of $K$. This lattice is therefore $K$-admissible, but not critical. Hence there exist lattices of smaller determinants. But this is impossible, since obviously $d(\Lambda^\times) = d(\Lambda)$.

By Theorem 11, any two points of $\Lambda$ on $C_1$, or any two such points on $C_2$, form a basis. Hence, if for brevity we write

$$(61) \quad Y = D(J), \quad \text{then} \quad \sqrt{3} \leq Y \leq 1.$$ 

For $K$ contains the circle $f_1 = 1$; further, $|\langle P, Q \rangle| \leq 1$ for any two points $P$ and $Q$ on $C_1$, or on $C_2$.

28. A sufficient condition for admissible lattices.

The construction of the critical lattices of $K$ makes use of

THEOREM 25. Suppose that the lattice $\Lambda$ of determinant

$$d(\Lambda) \geq \sqrt{3}$$

has a basis consisting of two points $P_1$, $P_2$ on $f_1 = 1$, and a second basis consisting of two points $P_3$, $P_4$ on $f_2 = 1$. Then $\Lambda$ is $K$-admissible.

Proof. It suffices to show that no lattice point $P \neq O$ is an inner point of $f_2 = 1$; the analogous result for $f_1 \leq 1$ is proved similarly.

Every point $P: (x, y)$ can be written as

$$P = uP_3 + vP_4, \quad \text{where} \quad u = \frac{\langle P, P_3 \rangle}{\langle P_3, P_4 \rangle}, \quad v = -\frac{\langle P, P_4 \rangle}{\langle P_3, P_4 \rangle}.$$
The new coordinates $u, v$ are integers if, and only if, $P$ is a lattice point. The result of replacing $x, y$ by $u, v$ is that $f_2$ takes the form

$$f_2(x, y) = f_2^*(u, v) = u^2 + 2uw + v^2,$$

since the two points $u = 1, v = 0$ and $u = 0, v = 1$ lie on $f_2^* = 1$. By the invariance property of the determinant of a quadratic form,

$$1 - s^2 = (P_3, P_4)^2 = d(\Lambda)^2 \geq \frac{3}{2},$$

so that

$$-\frac{1}{2} \leq s \leq \frac{1}{2}.$$

Hence $f_2^*$ is a reduced form†. Its minimum for integral $u, v$ not both zero is then 1, as asserted.

Henceforth let $S(J)$ be the set of lattices $\Lambda$ with the following properties:

(a) $\Lambda$ has a basis $P_1, P_2$ on $f_1 = 1$, and a basis $P_3, P_4$ on $f_2 = 1$.

(b) The determinant $d(\Lambda) \geq \sqrt{\frac{3}{2}}$.

We shall prove later that $S(J)$ has only a finite number of elements, say the lattices

$$\Lambda_1, \Lambda_2, ..., \Lambda_n.$$

By Theorem 25, these lattices are $K$-admissible; by Theorems 23 and 24, all critical lattices $\Lambda$ belong to $S(J)$. Hence

$$D(J) = \min_{r=1, 2, ..., n} d(\Lambda_r),$$

and so the critical lattices of $K$ are just those elements $\Lambda_r$ of $S(J)$ for which $d(\Lambda_r)$ assumes the minimum value $D(J)$.

29. Construction of the set $S(J)$.

Let $\Lambda$ be a lattice in $S(J)$. We may assume, without loss of generality, that the two bases

$$P_1: (x_1, y_1), P_2: (x_2, y_2) \quad \text{and} \quad P_3: (x_3, y_3), P_4: (x_4, y_4)$$

of $\Lambda$ satisfy the inequalities

$$\begin{align*}
(P_1, P_2) &> 0 \quad \text{and} \quad (P_3, P_4) > 0;
\end{align*}$$

† See footnote †, page 108.
(68) \[ d(\Lambda) = (P_1, P_2) = (P_3, P_4) = x_1 y_2 - x_2 y_1 = x_3 y_4 - x_4 y_3. \]

The inequalities (67) remain satisfied if the pair of points \( P_1, P_2 \) is replaced by one of the four pairs

\[ P_1, P_2, \quad P_2, -P_1, \quad -P_1, -P_2, \quad \text{or} \quad -P_2, P_1; \]

and if the pair of points \( P_3, P_4 \) is replaced by one of the four pairs

\[ P_3, P_4, \quad P_4, -P_3, \quad -P_3, -P_4, \quad \text{or} \quad -P_4, P_3. \]

This gives a set \( \Omega \) of \( 4 \times 4 = 16 \) pairs of bases of \( \Lambda \).

By the basis property and by (68), there are four integers \( a_1, \beta_1, a_2, \beta_2 \) such that

(69) \[ P_3 = a_1 P_1 + \beta_1 P_2, \quad P_4 = a_2 P_1 + \beta_2 P_2, \quad a_1 \beta_2 - a_2 \beta_1 = +1. \]

When the pair of bases \( P_1, P_2 \) and \( P_3, P_4 \) is replaced by one of the other pairs in \( \Omega \), then \( a_1, \beta_1, a_2, \beta_2 \) undergo certain permutations and changes of signs, for which I refer to the following table.

**The 16 elements of \( \Omega \).**

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Let a new system of rectangular coordinates $U, V$ be defined by

$$x = x_1 U - y_1 V, \quad y = y_1 U + x_1 V,$$

or conversely, since $x_1^2 + y_1^2 = 1$,

$$U = x_1 x + y_1 y, \quad V = -y_1 x + x_1 y.$$ 

In this system, $P_1$ and $P_2$ have the coordinates

$$U_1 = 1, \quad V_1 = 0 \quad \text{and} \quad U_2 = X = x_1 x_2 + y_1 y_2, \quad V_2 = Y = x_1 y_2 - x_2 y_1.$$ 

Here

$$X^2 + Y^2 = 1, \quad Y = d(\Lambda) > 0.$$ 

Further, by (69), the coordinates of $P_3$ and $P_4$ are given by

$$U_3 = a_1 + \beta_1 X, \quad V_3 = \beta_1 Y \quad \text{and} \quad U_4 = a_2 + \beta_2 X, \quad V_4 = \beta_2 Y.$$ 

Finally, if, as in §28, we introduce $u, v$ by (62), then

$$U = (a_1 + \beta_1 X) u + (a_2 + \beta_2 X) v, \quad V = \beta_1 X u + \beta_2 Y v,$$

and so, on solving for $u$ and $v$, we have

$$Y u = +\beta_2 Y U - (a_2 + \beta_2 X) V.$$ 

I refer to the last table for the changes of these numbers $a_1, \beta_1, a_2, \beta_2, X, Y, u, v$, when the pair of bases $P_1, P_2$ and $P_3, P_4$ is replaced by another pair in $\Omega$.

By §28, $f_2$ takes the form (63) in $u$ and $v$. By (64) and (72),

$$s = eX, \quad \text{where} \quad e = \pm 1.$$ 

An inspection of the table shows that it is always possible to choose the pair of bases $P_1, P_2$ and $P_3, P_4$ in $\Omega$ so that the following inequalities are satisfied:

$$X > 0, \quad s > 0, \quad a_1 > 0.$$ 

Therefore, in particular,

$$s = X.$$ 

Replace $u$ and $v$ by $U$ and $V$. Then $f_2$ changes into

$$f_2(x, y) = F_2(U, V) = AU^2 + 2BUV + CV^2.$$
where, by (63), (73), and (76),

\[
A = \beta_1^2 - 2\beta_1 \beta_2 X + \beta_2^2,
\]

(78) \[
YB = -\beta_1(a_1 + \beta_1 X) + X(\beta_2(a_1 + \beta_1 X) + \beta_1(a_2 + \beta_2 X)) - \beta_2(a_2 + \beta_2 X),
\]

\[
Y^2C = (a_1 + \beta_1 X)^2 - 2(a_1 + \beta_1 X)(a_2 + \beta_2 X)X + (a_2 + \beta_2 X)^2.
\]

Further, since the change from \(x, y\) to \(U, V\) is an orthogonal transformation,

\[
f_1(x, y) = F_1(U, V) = U^2 + V^2.
\]

Hence the simultaneous invariant

\[
J = A + C,
\]

so that, by (72) and (78),

(79) \[
(a_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 - J) - 2(a_1 - \beta_2)(a_{2} - \beta_1)X
\]

\[
- \{2(a_1 \beta_2 + a_2 \beta_1) - J\} X^2 = 0.
\]

For given \(J\), this is a quadratic equation for \(X\). It does not reduce to an identity, for then

\[
a_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = J, \quad 2(a_1 \beta_2 + a_2 \beta_1) = J;
\]

hence

\[
(a_1 - \beta_2)^2 + (a_2 - \beta_1)^2 = 0,
\]

and since \(a_1 \geq 0\), \(a_1 \beta_2 - a_2 \beta_1 = 1, \quad a_1 = \beta_2 = 1, \quad a_2 = \beta_1 = 0, \quad J = 2.
\]

This value of \(J\) was, however, excluded by § 25.

By the assumption \((b)\) in § 28, and by (72) and (75),

\[
0 \leq X \leq \frac{1}{4}.
\]

(80)

Suppose now, conversely, that (79) has a solution \(X\) satisfying these inequalities. Then the coefficients \(A, B, C\) of \(F\) are given by (78), with

(81)

\[
Y = |\sqrt{(1-X^2)}|.
\]

We further obtain the \((U, V)\)-coordinates of \(P_1, P_2, P_3, P_4\) from their expressions as functions of \(a_1, \beta_1, a_2, \beta_2, X, Y\). There remains the reduction of \(F_1(U, V)\) and \(F_2(U, V)\) to the normal form (54) by means of an orthogonal transformation (71); this problem is dealt with in the theory of conics. After this reduction, the \((x, y)\)-coordinates of \(P_1, P_2, P_3, P_4\) and so the lattice \(\Lambda\) are known.

Therefore, in order to construct all elements of \(S(J)\), it suffices to solve (79) with respect to \(X\). Here the coefficients \(a_1, \beta_1, a_2, \beta_2\) must take all integral values with

\[
a_1 \geq 0, \quad a_1 \beta_2 - a_2 \beta_1 = 1,
\]

for which both (79) and (80) can be satisfied.
30. The finiteness of \( S(J) \).

**Theorem 26.** The set \( S(J) \) has only a finite number of elements.

**Proof.** It suffices to show that the conditions (79) and (80) are solvable for at most a finite number of sets of integers \( a_1, \beta_1, a_2, \beta_2 \).

The equation (79) can be written as

\[
\Phi(X; a_1, \beta_1, a_2, \beta_2) = J,
\]

where

\[
\Phi(X; a_1, \beta_1, a_2, \beta_2) = \frac{(a_1^2 + \beta_1^2 + a_2^2 + \beta_2^2) - 2(a_1 - \beta_1)(a_2 - \beta_2)X - 2(a_1 \beta_2 + a_2 \beta_1)X^2}{1 - X^2}.
\]

This expression \( \Phi \) is a positive definite quadratic form in \( a_1, \beta_1, a_2, \beta_2 \); for it can be written as

\[
\Phi(X; a_1, \beta_1, a_2, \beta_2) = \frac{1}{1 - X^2} (a_1 - X^2 \beta_2 - X a_2 + X \beta_1)^2 + (1 + X^2) \left( \beta_2 + \frac{X}{1 + X^2} a_2 - \frac{X}{1 + X^2} \beta_1 \right)^2 + \frac{1}{1 + X^2} (a_2 + X^2 \beta_1)^2 + (1 - X^2) \beta_1^2.
\]

From this identity, by (80),

\[
\Phi(X; a_1, \beta_1, a_2, \beta_2) \geq (1 - X^2) \beta_1^2 \geq \frac{3}{2} \beta_1^2.
\]

Further, from the definition of \( \Phi \),

\[
\Phi(X; a_1, \beta_1, a_2, \beta_2) = \Phi(X; \beta_1, a_1, \beta_2, a_2) = \Phi(X; a_2, \beta_2, a_1, \beta_1) = \Phi(X; \beta_2, a_2, \beta_1, a_1).
\]

Hence \( \beta_1 \) may be replaced by \( a_1, \beta_1, a_2, \beta_2 \) in the last inequality, and so, by (83),

\[
\max (a_1^2, \beta_1^2, a_2^2, \beta_2^2) \leq \frac{4J}{3},
\]

which proves the assertion.

Let then \( \Lambda_\nu \) \( (\nu = 1, 2, \ldots, n) \) be the elements of \( S(J) \); let

\[
a^{(\nu)}_1, \beta^{(\nu)}_1, a^{(\nu)}_2, \beta^{(\nu)}_2 \quad (\nu = 1, 2, \ldots, n)
\]

be the sets of four integers; and let

\[
\Phi_\nu(X) = \Phi(X; a^{(\nu)}_1, \beta^{(\nu)}_1, a^{(\nu)}_2, \beta^{(\nu)}_2) \quad (\nu = 1, 2, \ldots, n)
\]

be the functions belonging to these lattices. The following table contains all functions \( \Phi_\nu \) which represent at least one value of \( J \) in \( 2 \leq J \leq 25 \) for an argument \( X \) in \( 0 \leq X \leq \frac{1}{2} \).
Table of all functions $\Phi$ which represent $J$ for $J \leq 25$.

<table>
<thead>
<tr>
<th>$\Phi(\xi)$</th>
<th>$(1-X^2)\Phi(X; a_1, b_1, a_2, b_2)$</th>
<th>$a_1$</th>
<th>$\beta_1$</th>
<th>$\sigma_1$</th>
<th>$\beta_2$</th>
<th>$a_2$</th>
<th>$\beta_1$</th>
<th>$\sigma_2$</th>
<th>$\beta_2$</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>$2-2X^2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$-4X+2X^2$</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>$6-8X+2X^2$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>5</td>
<td>$7-12X+6X^2$</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>6</td>
<td>$3+4X+2X^2$</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>$1-2X^2$</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>$6-6X^2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>$11-12X+2X^2$</td>
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<td>1</td>
<td>-1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>10</td>
<td>$15-24X+10X^2$</td>
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<td>-3</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>11</td>
<td>$18-32X+14X^2$</td>
<td>1</td>
<td>-2</td>
<td>2</td>
<td>-3</td>
<td>3</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>12</td>
<td>$6+8X+2X^2$</td>
<td>0</td>
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<td>-2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>13</td>
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<td>1</td>
<td>1</td>
<td>-3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>$15-4X-10X^2$</td>
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<td>3</td>
<td>1</td>
<td>2</td>
</tr>
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<td>$18-16X+2X^2$</td>
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<td>-1</td>
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<td>4</td>
</tr>
<tr>
<td>16</td>
<td>$27-40X+14X^2$</td>
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<td>-4</td>
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<td>-3</td>
<td>1</td>
<td>-1</td>
<td>4</td>
<td>-3</td>
</tr>
<tr>
<td>17</td>
<td>$38-72X+34X^2$</td>
<td>2</td>
<td>-3</td>
<td>3</td>
<td>-4</td>
<td>4</td>
<td>-3</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>18</td>
<td>$7+12X+6X^2$</td>
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<td>2</td>
<td>-1</td>
<td>-1</td>
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<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>19</td>
<td>$15+4X-10X^2$</td>
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<td>3</td>
<td>1</td>
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<td>2</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>$18-14X^2$</td>
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<td>3</td>
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<td>2</td>
<td>2</td>
<td>-3</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>21</td>
<td>$34-48X+18X^2$</td>
<td>2</td>
<td>-5</td>
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<td>-2</td>
<td>2</td>
<td>-1</td>
<td>5</td>
<td>-2</td>
</tr>
<tr>
<td>22</td>
<td>$39-60X+22X^2$</td>
<td>1</td>
<td>-3</td>
<td>2</td>
<td>-5</td>
<td>1</td>
<td>-2</td>
<td>3</td>
<td>-5</td>
</tr>
<tr>
<td>23</td>
<td>$47-84X+38X^2$</td>
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<td>-5</td>
<td>2</td>
<td>-3</td>
<td>3</td>
<td>-2</td>
<td>5</td>
<td>-3</td>
</tr>
<tr>
<td>24</td>
<td>$11+12X+2X^2$</td>
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<td>1</td>
<td>-1</td>
<td>-3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>25</td>
<td>$18-2X^2$</td>
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<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>26</td>
<td>$27-20X+2X^3$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>5</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-5</td>
</tr>
<tr>
<td>27</td>
<td>$27-12X-14X^2$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>28</td>
<td>$43-60X+18X^2$</td>
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<td>-5</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-1</td>
<td>5</td>
<td>-4</td>
</tr>
<tr>
<td>29</td>
<td>$66-128X+62X^2$</td>
<td>3</td>
<td>-4</td>
<td>4</td>
<td>-5</td>
<td>5</td>
<td>-4</td>
<td>4</td>
<td>-3</td>
</tr>
</tbody>
</table>

Excluded case.

Singular lattices.
As this table shows, there are in general two, three, or four systems of integers \(a_1^{(\nu)}, \beta_1^{(\nu)}, a_2^{(\nu)}, \beta_2^{(\nu)}\) belonging to the same function \(\Phi_\nu\) and so also an equal number of lattices \(\Lambda_\nu\).† It is easily seen that if there are different critical lattices belonging to the same function \(\Phi_\nu\), then these are transformed into one another by the group \(G\) of order \(4\) generated by the following two affine transformations:

**The symmetry in the y-axis,**

A: \(x \to -x, \ y \to y\).

The interchange of \(f_1 = 1\) and \(f_2 = 1\),

B: \(x \to \lambda^{-1}y, \ y \to \lambda^1x\).

For A replaces the integers \(a_1, \beta_1, a_2, \beta_2\) by

\(\epsilon \beta_2, \ \epsilon a_2, \ \epsilon \beta_1, \ \epsilon a_1\),

where \(\epsilon = \pm 1\) is such that \(\epsilon \beta_2 \geq 0\), and B replaces them by

\(a_1, -a_2, -\beta_1, \ \beta_2\).  

From now on, two critical lattices are considered as equivalent if they are related by an element of this group \(G\); equivalent lattices belong to the same function \(\Phi_\nu\).

31. The value of \(D(J)\) for \(2 \leq J \leq 25\).

By formula (66) in § 28,

\[D(J) = \min_{\nu=1, 2, \ldots, n} d(\Lambda_\nu).\]

Hence, if

\[Y = D(J), \ X = |\sqrt{(1-Y^2)}|, \text{ and } Y_\nu = d(\Lambda_\nu), \ X_\nu = |\sqrt{(1-Y_\nu^2)}|,\]

then

\[(85) \quad \Phi_\nu(X_\nu) = J, \ 0 \leq X_\nu \leq \frac{1}{2},\]

\[(86) \quad X = \max_{\nu=1, 2, \ldots, n} X_\nu.\]

† Two systems of integers

\(0, 1, -1, \beta^{(\nu)}\) and \(0, -1, 1, -\beta^{(\nu)}\)

are interchanged by elements of \(\Omega\) (§ 29) and generate the same lattice.
By a study of the last table I find that for every \( J \) in \( 2 \leq J \leq 25 \) and for every \( \Phi_\ast \) there is at most one solution \( X_\ast \) of (85). Further, most of these solutions \( X_\ast \) can be ignored for the following reasons.

The rows of the table have been arranged in sets of functions

\[(1-X^2)\Phi_\ast(X)\]

so that \( \Phi_\ast(\frac{1}{2}) \) is the same in each set. It was also found possible to arrange the rows according to increasing values of these functions for variable values of \( X \); e.g., in the second set,

\[
\frac{2+2X^2}{1-X^2} < \frac{3-2X^2}{1-X^2} < \frac{6-8X+2X^2}{1-X^2} < \frac{7-12X+8X^2}{1-X^2} \quad \text{for} \quad 0 \leq X \leq \frac{1}{2}.
\]

Hence, for a given value of \( J \) in \( 2 \leq J \leq 25 \), the maximum \( X = X_\ast \) belongs to one of those 11 equations

\[\Phi_\ast(X_\ast) = J\]

in which the function \( \Phi_\ast \) is either at the beginning or at the end of one of the 6 sets of rows of the table. There is no difficulty in deciding which is the largest of these solutions \( X_\ast \). The result depends on the value of \( J \), and is given in the following table. This table further contains the minimum determinant

\[D(J) = \Delta(K)\]

and the corresponding critical lattice\(^\dagger\).

In the table, the numbers \( \sigma_k \) are defined thus:

\[\sigma_0 = 2, \quad \sigma_1 = \frac{14}{2}, \quad \sigma_2 = \frac{3}{2}, \quad \sigma_3 = 14, \quad \sigma_4 = \frac{5}{2}, \quad \sigma_5 = \frac{7}{2};\]

and \( J_n \) is defined thus

\[J_1 = \frac{3}{5}, \quad J_2 = \frac{3+14\sqrt{3}}{6}, \quad J_3 = 10, \quad J_4 = \frac{178+576\sqrt{14}}{143}, \quad J_5 = \frac{63+88\sqrt{7}}{14}.
\]

\(\dagger\) If there exist several critical lattices, then they are all equivalent to the one given, except when \( J \) is one of the numbers \( \sigma_\ast \) or \( J_\ast \).
### $D(J)$ and critical lattices for $2 \leq J \leq 25.$

<table>
<thead>
<tr>
<th>No.</th>
<th>Interval.</th>
<th>$(1 - X^2) Y =$</th>
<th>$X =$</th>
<th>$D(J) = Y =$</th>
<th>Critical lattice.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sigma_6 &lt; J &lt; J_1$</td>
<td>$3 - 4X + 2X^2$</td>
<td>$\frac{2 - (J^2 - J - 2)}{J + 2}$</td>
<td>$\frac{(5J + 2 + 4(J^2 - J - 2)^2)}{J + 2}$</td>
<td>$P_1 = P_3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$P_4 = -P_1 + P_3$.</td>
</tr>
<tr>
<td>2</td>
<td>$J_1 &lt; J &lt; \sigma_1$</td>
<td>$2 + 2X^2$</td>
<td>$\frac{(J - 2)^4}{(J + 2)^4}$</td>
<td>$2(J + 2)^{-4}$</td>
<td>$P_4 = P_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$P_3 = P_1$.</td>
</tr>
<tr>
<td>3</td>
<td>$\sigma_1 &lt; J &lt; J_2$</td>
<td>$7 - 12X + 6X^2$</td>
<td>$\frac{6 - (J^2 - J - 6)^4}{J + 6}$</td>
<td>$\frac{(13J + 6 + 12(J^2 - J - 6)^4)}{J + 6}$</td>
<td>$P_4 = P_1 - 2P_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$P_3 = P_1 - P_2$.</td>
</tr>
<tr>
<td>4</td>
<td>$J_2 &lt; J &lt; \sigma_2$</td>
<td>$3 + 4X + 2X^2$</td>
<td>$\frac{-2 + (J^2 - J - 2)^4}{J + 2}$</td>
<td>$\frac{(5J + 2 + 4(J^2 - J - 2)^2)}{J + 2}$</td>
<td>$P_3 = P_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$P_4 = -P_1$.</td>
</tr>
<tr>
<td>5</td>
<td>$\sigma_2 &lt; J &lt; J_3$</td>
<td>$18 - 32X + 14X^2$</td>
<td>$\frac{J - 18}{J + 14}$</td>
<td>$8(J - 2)^4$</td>
<td>$P_3 = P_1 - 2P_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$P_4 = 2P_1 - 3P_2$.</td>
</tr>
<tr>
<td>6</td>
<td>$J_3 &lt; J &lt; \sigma_2$</td>
<td>$6 + 8X + 2X^2$</td>
<td>$\frac{J - 6}{J + 2}$</td>
<td>$4(J - 2)^4$</td>
<td>$P_4 = P_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$P_3 = -P_1 - 2P_2$.</td>
</tr>
<tr>
<td>7</td>
<td>$\sigma_3 &lt; J &lt; J_4$</td>
<td>$38 - 72X + 34X^2$</td>
<td>$\frac{-J - 38}{J + 34}$</td>
<td>$12(J - 2)^4$</td>
<td>$P_3 = 2P_1 - 3P_2$</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>$P_4 = 3P_1 - 4P_2$.</td>
</tr>
<tr>
<td>8</td>
<td>$J_4 &lt; J &lt; \sigma_4$</td>
<td>$7 + 12X + 6X^2$</td>
<td>$\frac{-6 + (J^2 - J - 0)^2}{J + 6}$</td>
<td>$\frac{(13J + 6 + 12(J^2 - J - 6)^2)}{J + 6}$</td>
<td>$P_3 = P_1 + 2P_2$</td>
</tr>
<tr>
<td></td>
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<td></td>
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<td></td>
<td>$P_4 = -P_1 - P_2$.</td>
</tr>
<tr>
<td>9</td>
<td>$\sigma_4 &lt; J &lt; J_5$</td>
<td>$47 - 84X + 38X^2$</td>
<td>$\frac{42 - (J^2 - 9J - 22)^2}{J + 38}$</td>
<td>$\frac{(85J - 298 + 84(J^2 - 9J - 22)^2)}{J + 38}$</td>
<td>$P_3 = 3P_1 - 3P_3$</td>
</tr>
<tr>
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<td></td>
<td>$P_4 = 2P_1 - 3P_2$.</td>
</tr>
<tr>
<td>10</td>
<td>$J_5 &lt; J &lt; \sigma_5$</td>
<td>$11 + 12X + 2X^2$</td>
<td>$\frac{-6 + (J^2 - 9J + 14)^2}{J + 2}$</td>
<td>$\frac{(13J - 46 + 12(J^2 - 9J + 14)^2)}{J + 2}$</td>
<td>$P_3 = P_1$</td>
</tr>
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<td></td>
<td></td>
<td>$P_4 = -P_1 - 3P_2$.</td>
</tr>
<tr>
<td>11</td>
<td>$\sigma_5 &lt; J &lt; 25$</td>
<td>$66 - 128X + 62X^2$</td>
<td>$\frac{-J - 66}{J + 62}$</td>
<td>$\frac{16(J - 2)^4}{J + 62}$</td>
<td>$P_3 = 3P_1 - 4P_3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$P_4 = 4P_1 - 5P_3$.</td>
</tr>
</tbody>
</table>

† Singular lattice.  
‡ These values of $X$ and $Y$ remain true for $\sigma_6 < J < 25$.  

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In the intervals No. 1–11 of the table, the functions $X = X(J)$ and $Y = Y(J)$ behave in the following manner:

- $X$ is steadily increasing and $Y$ is steadily decreasing in the intervals No. 2, 4, 6, 8, 10.
- $X$ is steadily decreasing and $Y$ is steadily increasing in the intervals No. 1, 3, 5, 7, 9, 11.

Further,

$$X = \frac{1}{4}, \quad Y = \frac{\sqrt{3}}{2} \quad \text{for} \quad J = \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5,$$

and

- $X = \frac{1}{2}$, $Y = \frac{\sqrt{15}}{4}$ for $J = J_1$,
- $X = 2 - \sqrt{3}$, $Y = \sqrt{\{4 \sqrt{(3)}-6\}}$ for $J = J_2$,
- $X = \frac{4}{3}$, $Y = \frac{\sqrt{2}}{3}$ for $J = J_3$,
- $X = \frac{21-4 \sqrt{14}}{14}$, $Y = \sqrt{(\frac{24 \sqrt{(14)}-67}{28})}$ for $J = J_4$,
- $X = \frac{4-\sqrt{7}}{3}$, $Y = \sqrt{(\frac{8 \sqrt{(14)}-14}{9})}$ for $J = J_5$.

The interval No. 2 is particularly interesting, since here $K$ has only a single critical lattice, and this is singular. At the lower end $J = \frac{3}{4}$ of this interval, $K$ has this singular lattice, and also the regular lattice

$$P_3 = P_2, \quad P_4 = -P_1 + P_2,$$

and the lattice symmetrical to it in the $y$-axis.

The table shows that the critical lattices of $K$ have 2, 3, 4, 5, or 6 pairs of symmetrical points on $C$, depending on the value of $J$.

The general law of the function $D(J)$ seems to be very complicated. By the table, the graph of $Y = D(J)$ is a saw-like curve for $2 \leq J \leq 25$, and possibly for all values of $J$. In the intervals No. 5, 6, 7, and 11, $D(J)$ takes a surprisingly simple form.
One can show that \( \frac{\sqrt{3}}{2} \leq D(J) \leq \frac{\sqrt{15}}{4} \) for all values of \( J \), and that

\[
\lim_{J \to \infty} D(J) = \frac{\sqrt{3}}{2};
\]

this limit equation was communicated to me by P. Erdős.

I remark finally that the problem and result of this chapter can be extended to a pair of positive definite Hermitian forms; but then the proof is preferably based on the geometrical theory of Picard's group.

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Manchester, 13.