ON REDUCED POSITIVE DEFINITE QUATERNARY QUADRATIC FORMS

BY

KURT MAHLER
(Manchester).

According to Minkowski's definition, a positive definite quadratic form in \( n \) variables with real coefficients

\[
f(x) = \sum_{h=1}^{n} a_{hh} x_h^2
\]

is called reduced, if for \( h = 1, 2, \ldots, n \)

\[
f(x) \geq a_{hh} \text{ for all integers } x_1, \ldots, x_h \text{ such that } (x_h, x_{h+1}, \ldots, x_n) = 1,
\]

and also

\[
a_{12} \geq 0, a_{23} \geq 0, \ldots, a_{n-1,n} \geq 0.
\]

Minkowski, using a method of Hermite, proved that there is a constant \( c_n > 0 \) depending only on \( n \), such that for reduced forms\(^2\)

\[
a_{11}a_{22} \cdots a_{nn} \geq c_n D,
\]

where \( D \) is the determinant of \( f(x) \). For the lowest values of \( n \), the smallest value of this constant is

\[
c = \frac{1}{2}, c_3 = 2, c_4 = 4.
\]

The first result is classic, the second one due to Gauss (who proved it for Seeber's definition of a reduced ternary form, which is nearly identical with the case \( n = 3 \) of Minkowski's definition). I prove here the formula \( c_4 = 4 \), which seems to be new.\(^*)

My proof is derived from one of Minkowski for \( c_3 = 2 \).\(^3\)

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\(^1\) Ges. Abh. II, 88—100.

\(^2\) I.e., by see also my note, Quart. Journ. 9 (1938), 259—263.

\(^3\) Gauss, Werke II.

\(^*\) Addition September 1946. Compare the paper by R. Remak, Proc. Royal Acad. Amsterdam, 44, (1931), 1071—1076, where a similar method is used to study pseudoreduction of quadratic forms.

\(^*\) Ges. Abh. II.
and is based on the following theorem of Korkine and Zolotareff 9):

"For every positive definite quaternary quadratic form \( f(x) \) of determinant \( D_0 \), there exists a lattice point \( x \neq 0 \) such that

\[
f(x) \leq \sqrt[4]{4D_0}
\]

with equality if and only if \( f(x) \) is equivalent to the form \( \sqrt[4]{4D_0} x_0^{\delta_0} \), where \( \delta_0(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1 + x_2 + x_3)x_4 \). (1)

Proof of the inequality \( a_1a_2a_3a_4 \leq 4D \) for \( n = 4 \).

I use the vector notation; lower indices denote the different coordinates, upper ones different points.

Let

\[
x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)}) \quad (k = 1, 2, 3, 4)
\]

be four lattice points such that \( x^{(1)} \neq 0 \), and \( f(x^{(1)}) = A_{11} \) is the minimum of \( f(x) \) in all lattice points \( x \neq 0 \); and such that for \( k = 2, 3, \) and \( 4, \) \( x^{(k)} \) is linearly independent of

\[
x^{(1)}, \ldots, x^{(k-1)}, \text{ and } f(x^{(k)}) = A_{kk}
\]

is the minimum of \( f(x) \) for all lattice points \( x \) which are linearly independent of \( x^{(1)}, \ldots, x^{(k-1)} \). The four points \( x^{(k)} \) are therefore linearly independent, and their determinant

\[
d = \det x^{(k)}_{k,h = 1,2,3,4}
\]

is a non-vanishing integer.

An arbitrary point \( x = (x_1, x_2, x_3, x_4) \) can be written as

\[
x = \sum_{k=1}^{4} X_k x^{(k)},
\]

where the \( X_k \) are real numbers; let \( X = (X_1, X_2, X_3, X_4) \) be the point with these numbers as its coordinates. Then change of \( x \) into \( X \) is an integral linear transformation of determinant \( d \), namely

\[
x^{(k)} = \sum_{h=1}^{4} x^{(h)} x_{k,h} \quad (h = 1, 2, 3, 4) \quad (2)
\]

If \( X \) is a lattice point, then so is \( x \); the converse need not hold. The transformation (2) changes \( f(x) \) into a new quadratic form

\[
F(X) = f \left( \sum_{k=1}^{4} X_k x^{(k)} \right) = \sum_{k,h=1}^{4} A_{kk} X_k X_h,
\]

where the \( A_{kk} \) are the numbers as defined before. Since, if necessary, we may replace \( x^{(k)} \) by \(-x^{(k)}\), we can assume that

\[
A_{11} \geq 0, \quad A_{22} \geq 0, \quad A_{33} \geq 0.
\quad (3)
\]

By the definition of the lattice points \( x^{(k)} \),

\[
\begin{align*}
A_{11} & \quad \sum_{j=1}^{4} X_j^2 > 0, \\
A_{22} & \quad \sum_{j=1}^{4} X_j^2 > 0, \\
A_{33} & \quad \sum_{j=1}^{4} X_j^2 > 0, \\
A_{44} & \quad X_4^2 > 0.
\end{align*}
\quad (4)
\]

The same inequalities with \( F(X) \) instead of \( f(x) \) hold if \( X \) is a lattice point; hence \( F(X) \) is a reduced form.

We can write \( F(X) \) as a sum of squares of linear forms,

\[
\frac{1}{2} F(X) = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2.
\quad (5)
\]

such that \( \varepsilon_3 \) contains only \( X_1, \ldots, X_4 \); except for changes of sign, this representation is unique. If we replace \( X \) by \( x \) according to (2), then (5) is transformed into the analogous representation

\[
f(x) = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2
\quad (6)
\]

of \( f(x) \) as a sum of squares of linear forms in the \( x \)'s. In this representation,

\[\xi_h \text{ vanishes if } X_h = X_{h+1} = \ldots = X_4 = 0.\]

Let

\[
g(x) = \frac{\xi_1^2}{A_{11}} + \frac{\xi_2^2}{A_{22}} + \frac{\xi_3^2}{A_{33}} + \frac{\xi_4^2}{A_{44}}
\quad (7)
\]
be a new quadratic form of determinant

\[ D' = \frac{D}{A_{11}A_{22}A_{33}A_{44}}. \]  

(8)

Since by the definition of the minima \( A_{kk} \)

\[ 0 < A_{11} \leq A_{22} \leq A_{33} \leq A_{44}, \]

we get from (4) for lattice points \( x \) that

\[ g(x) = \mathbf{\frac{x^2}{A_{11}} + \frac{x^2}{A_{22}} + \frac{x^2}{A_{33}} + \frac{x^2}{A_{44}}} \geq \mathbf{1}, \]

if \( x_1 \neq 0, x_2 = x_3 = x_4 = 0; \)

\[ g(x) = \mathbf{\frac{x^2}{A_{11}} + \frac{x^2}{A_{22}} + \frac{x^2}{A_{33}} + \frac{x^2}{A_{44}}} \geq \mathbf{1}, \]

if \( x_2 \neq 0, x_3 = x_4 = 0; \)

\[ g(x) = \mathbf{\frac{x^2}{A_{11}} + \frac{x^2}{A_{22}} + \frac{x^2}{A_{33}} + \frac{x^2}{A_{44}}} \geq \mathbf{1}, \]

if \( x_3 \neq 0, x_4 = 0; \)

\[ g(x) = \mathbf{\frac{x^2}{A_{11}} + \frac{x^2}{A_{22}} + \frac{x^2}{A_{33}} + \frac{x^2}{A_{44}}} \geq \mathbf{1}, \]

if \( x_4 \neq 0. \)  

(9)

Therefore for every lattice point \( x, \)

\[ g(x) \geq 1. \]

By the theorem of Korkine and Zolotareff, this implies \( 1 \leq \sqrt[4]{D'} \) and therefore by (8),

\[ A_{11}A_{22}A_{33}A_{44} \leq 4D'. \]  

(10)

We consider firstly the case that the sign of equality holds in (10), so that \( g(x) \) has the determinant \( D' = 4D = \frac{1}{4}. \)

By the theorem of Korkine and Zolotareff, \( g(x) \) must therefore be equivalent to

\[ \phi_0(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1 + x_2 + x_3) \cdot x_4. \]

Hence there are 12 essentially different lattice points \(^6\)

\(^6\) The equation \( \phi_0(x) = 1 \) has the twelve solutions \((0000), (0010), (0001), (100-1), (010-1), (001-1), (110-1), (011-1), (101-1), (111-1), (111-2), \) and twelve further one derived from these by changing all signs.

\[ \xi \in \{ \xi_1(x), \xi_4(x) \} \]

in (6) vanishes identically. Hence at least one of the four numbers

\[ \xi_k(\phi^{(k)}) \quad (k = 1, 2, 3, 4), \]

say the number \( \xi_1(\phi^{(k)}) \), and at least one of the four numbers

\[ \xi_k(\phi^{(k)}) \quad (k = 1, 2, 3, 4), \]

say the number \( \xi_4(\phi^{(k)}) \), is different from zero. If \( h_1 = h_2 \), then

\[ \xi_1(\phi^{(k)}) \neq 0, \xi_4(\phi^{(k)}) \neq 0 \quad \text{for} \quad h_0 = h_1 = h_2. \]

We prove that if there is no index \( k = 1, 2, 3 \) or 4 such that both \( \xi_1(\phi^{(k)}) \) and \( \xi_4(\phi^{(k)}) \) are different from zero, there is still at least one index \( h_0 \) in the interval \( 1 \leq h_0 \leq 12 \) such that

\[ \xi_1(\phi^{(h)}) \neq 0, \xi_4(\phi^{(h)}) \neq 0. \]  

(11)

For reasons of symmetry, it obviously suffices to consider the cases that \( h_1 = 1, h_2 = 4, or \) that \( h_1 = 1, h_2 = 2. \) In the first case

\[ \xi_1(\phi^{(1)}) \neq 0, \xi_4(\phi^{(1)}) = 0; \quad \xi_1(\phi^{(4)}) = 0, \xi_4(\phi^{(4)}) \neq 0, \]

and therefore

\[ \xi_1(\phi^{(4)}) = \xi_1(\phi^{(1)}) - \xi_4(\phi^{(4)}) \neq 0, \xi_4(\phi^{(4)}) = \xi_4(\phi^{(1)}) - \xi_4(\phi^{(4)}) \neq 0. \]

In the second case

\[ \xi_1(\phi^{(1)}) \neq 0, \xi_4(\phi^{(1)}) = 0; \quad \xi_1(\phi^{(4)}) = 0, \xi_4(\phi^{(4)}) \neq 0, \]

and furthermore without loss of generality
\[ \xi_1 (\phi^{(4)}) = \xi_4 (\phi^{(4)}) = 0; \]
hence
\[ \xi_1 (\phi^{(b)}) - \xi_1 (\phi^{(u)}) + \xi_1 (\phi^{(b)}) - \xi_1 (\phi^{(u)}) \neq 0, \]
\[ \xi_4 (\phi^{(b)}) = \xi_4 (\phi^{(b)}) + \xi_4 (\phi^{(b)}) - \xi_4 (\phi^{(b)}) \neq 0. \]

The lattice point \( \phi^{(b)} \) in (11) satisfies the further inequality
\[ X_4 = X_4 (\phi^{(b)}) \neq 0, \]
since \( \frac{X_4^2}{\xi_4^2} \) is a non-vanishing constant. Hence by (9)
\[
g (\phi^{(b)}) = 1 = \frac{\xi_1 (\phi^{(b)})^2}{A_{11}} + \frac{\xi_2 (\phi^{(b)})^2}{A_{22}} + \frac{\xi_3 (\phi^{(b)})^2}{A_{33}} + \frac{\xi_4 (\phi^{(b)})^2}{A_{44}}
\geq \frac{\xi_1 (\phi^{(b)})^2 + \xi_2 (\phi^{(b)})^2 + \xi_3 (\phi^{(b)})^2 + \xi_4 (\phi^{(b)})^2}{A_{44}} = f (\phi^{(b)}) \geq 1, \]
and since \( 0 < A_{11} \leq A_{22} \leq A_{33} \leq A_{44} \), we must have
\[ A_{11} = A_{22} = A_{33} = A_{44} = \sqrt{4D}. \]
Therefore \( f (x) \) is equivalent to the form
\[ \sqrt{4D} \varphi_q (x). \]
Hence, if \( f (x) \) itself is reduced, then \( a_{11} = a_{22} = a_{33} = a_{44} = \sqrt{4D}. \)
and the assertion is proved . . . .

Secondly, let (10) be true with the sign \( "<" \). The form \( F (X) \) has the determinant \( Dd^2 \); therefore by a well known property of positive definite quadratic forms
\[ Dd^2 = A_{11} A_{22} A_{33} A_{44}, \]
and by (10),
\[ Dd^2 < 4D, \quad d^2 < 4, \quad d = \pm 1, \]
since \( d \) is a non-vanishing integer. Hence now the reduced form \( F (X) \) is equivalent to \( f (x) \); therefore, if \( f (x) \) is also reduced, then the statement follows at once, since \( a_{11} = A_{11}, \ a_{22} = A_{22}; \ a_{33} = A_{33}; \ a_{44} = A_{44}. \)

1) Two equivalent reduced forms \( f (x) = \sum_{h,k=1}^{n} a_{hk} x_h x_k \) and \( F (X) = \sum_{h=1}^{n} A_{kk} X_k X_k \)
satisfy the equations
\[ a_{kk} = A_{kk} \quad (k = 1, 2, \ldots, n), \]
since both are lowest forms.

Mathematics Department, 24th May, 1940.
Manchester University.

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