On Dyson's improvement of the Thue-Siegel theorem

BY

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Two years ago, F. J. DYSON proved the following result 1):
“If \( \xi \) is an algebraic number of degree \( n \geq 2 \), if \( \mu \) is a positive number, and if there are infinitely many rational numbers \( \frac{p}{q} \) such that

\[
p, q \text{ are integers}, \quad q \geq 1, \quad \left| \xi - \frac{p}{q} \right| < q^{-\mu},
\]

then

\[
\mu \leq \sqrt{2n}.
\]

This result is stronger than that of C. L. SIEGEL 2), namely

\[
\mu \leq \min_{s=1,2,\ldots,n-1} \left( \frac{n}{s+1} + s \right) < 2 \sqrt{n}.
\]

DYSON obtained his improved inequality by means of a new method for studying the zero points of a polynomial in two variables. As his own proof is given in a somewhat involved form, I present in this paper a simplified proof for his main lemma (Theorem 1). Moreover, since this proof is purely algebraic, I deal always with the case of an arbitrary constant field of characteristic zero. This restriction is a natural one, since neither Theorem 1, nor the Thue–Siegel theorem, hold generally for fields of positive characteristic.

P.S. Since the time earlier this year when I wrote the present paper, a new proof of Dyson’s result has been published by TH. SCHNEIDER 3). This proof applies the deeper arithmetical properties of divisibility and may prove more powerful 4).

[1] In this paper, \( K \) denotes a fixed field of characteristic zero; \( K[x], K[y], \) and \( K[x, y] \), are the rings of all polynomials in \( x, \) in \( y, \) or in \( x \) and \( y, \) respectively, with coefficients in \( K; \) and \( K(x) \) denotes the field of all rational functions in \( x \) with coefficients in \( K. \) The terms “dependent” and “independent” always mean, “linearly dependent” and “linearly independent” over \( K. \)

4) Still another proof and a generalization of Dyson’s theorem was given by A. O. GELFOND (Vestnik MGU 9, 3 (1948)), but I have not seen his paper.
We define differentiation in $K(x)$ in the usual formal way. Let $u_0(x), u_1(x), \ldots, u_{l-1}(x)$ be a finite set of elements of $K(x)$; the determinant

$$\left| \frac{d^n u_\lambda(x)}{dx^n} \right|_{\lambda, \mu = 0, 1, \ldots, l-1}$$

is then called the Wronski determinant of these elements and is denoted by

$$\langle u_0, u_1, \ldots, u_{l-1} \rangle.$$ 

One easily verifies that if $\varphi(x)$ is any further element of $K(x)$, then

$$\langle \varphi u_0, \varphi u_1, \ldots, \varphi u_{l-1} \rangle = \varphi(x)^l \langle u_0, u_1, \ldots, u_{l-1} \rangle.$$

**Lemma 1:** The Wronski determinant of any finite number of elements of $K(x)$ vanishes identically in $x$ if, and only if, these elements are dependent.

**Proof:** If

$$\sum_{\lambda = 0}^{l-1} c_\lambda u_\lambda(x) \equiv 0, \quad \text{where } c_\lambda \in K,$$

then

$$\sum_{\lambda = 0}^{l-1} c_\lambda \frac{d^n u_\lambda(x)}{dx^n} \equiv 0 \quad (\mu = 0, 1, \ldots, l-1),$$

whence $\langle u_0, u_1, \ldots, u_{l-1} \rangle \equiv 0$.

Next assume that $\langle u_0, u_1, \ldots, u_{l-1} \rangle \equiv 0$; we must show that $u_0(x), u_1(x), \ldots, u_{l-1}(x)$ are dependent. This assertion is obvious for $l = 1$; assume it has already been proved for all systems of less than $l$ rational functions. We may exclude the case that $u_0(x) \equiv 0$ since then the Wronski determinant certainly vanishes. Hence

$$u_0(x)^{-1} \langle u_0, u_1, \ldots, u_{l-1} \rangle = \left\langle 1, \frac{u_1}{u_0}, \frac{u_2}{u_0}, \ldots, \frac{u_{l-1}}{u_0} \right\rangle =$$

$$= \left\langle \frac{d(u_1/u_0)}{dx}, \frac{d(u_2/u_0)}{dx}, \ldots, \frac{d(u_{l-1}/u_0)}{dx} \right\rangle \equiv 0.$$

Therefore, by the induction hypothesis, there exist $l-1$ elements $c_1, c_2, \ldots, c_{l-1}$ of $K$ not all zero such that

$$c_1 \frac{d(u_1/u_0)}{dx} + c_2 \frac{d(u_2/u_0)}{dx} + \ldots + c_{l-1} \frac{d(u_{l-1}/u_0)}{dx} \equiv 0.$$

Since the characteristic of $K$ is zero, this implies that

$$c_0 + c_1 \frac{u_1(x)}{u_0(x)} + c_2 \frac{u_2(x)}{u_0(x)} + \ldots + c_{l-1} \frac{u_{l-1}(x)}{u_0(x)} \equiv 0$$

for some element $c_0$ of $K$, whence the assertion.
Let now \( u_0(x), u_1(x), \ldots, u_{l-1}(x) \) be a finite set of independent polynomials in \( K[x] \), and assume that \( u_0(x) \) is of the highest degree amongst these, the degree \( d_0 \), say. Then constants \( c_1, c_2, \ldots, c_{l-1} \) in \( K \) can be found such that
\[
u^{(1)}_\lambda(x) = c_\lambda u_0(x) + u_\lambda(x) \quad (\lambda = 1, 2, \ldots, l-1)
\]
are all of degree less than \( d_0 \). Assume that \( u_1(x) \) is of highest degree, \( d_1 \) say, amongst these \( l - 1 \) polynomials. Then constants \( c^{(1)}_2, c^{(1)}_3, \ldots, c^{(1)}_{l-1} \) in \( K \) can be found such that the \( l - 2 \) polynomials
\[
u^{(2)}_\lambda(x) = c^{(1)}_\lambda u^{(1)}_1(x) + u^{(1)}_\lambda(x) \quad (\lambda = 2, 3, \ldots, l-1)
\]
are all of degree less than \( d_1 \). Assume that \( u^{(2)}_2(x) \) is of highest degree, \( d_2 \) say, amongst these polynomials. Then constants \( c^{(2)}_3, c^{(2)}_4, \ldots, c^{(2)}_{l-1} \) can be found such that the \( l - 3 \) polynomials
\[
u^{(3)}_\lambda(x) = c^{(2)}_\lambda u^{(2)}_2(x) + u^{(2)}_\lambda(x) \quad (\lambda = 3, 4, \ldots, l-1)
\]
are all of degree less than \( d_2 \). Continuing in this way, we obtain a set of \( l \) polynomials
\[
u_0(x), \nu^{(1)}_1(x), \nu^{(2)}_2(x), \ldots, \nu^{(l-1)}_{l-1}(x)
\]
of degrees
\[d_0, d_1, d_2, \ldots, d_{l-1}\]
respectively, where
\[d_0 > d_1 > d_2 > \ldots > d_{l-1}.
\]
By the construction, each polynomial \( \nu^{(i)}_\lambda(x) \) differs from \( \nu_\lambda(x) \) only by a linear expression in \( u_0(x), u_1(x), \ldots, u_{l-1}(x) \) with coefficients in \( K \). Hence, by a simple property of determinants, the identity
\[
\langle u_0, u_1, \ldots, u_{l-1} \rangle = \langle u_0, u^{(1)}_1, \ldots, u^{(l-1)}_{l-1} \rangle
\]
holds.

**Lemma 2:** Let \( u_0(x), u_1(x), \ldots, u_{l-1}(x) \) be polynomials in \( K[x] \) of degrees not greater than \( d \). Then the Wronski determinant
\[
\langle u_0, u_1, \ldots, u_{l-1} \rangle
\]
is a polynomial of degree not greater than \( l(d - l + 1) \).

**Proof:** It suffices to prove the assertion when the polynomials are independent. The polynomials
\[
u_0(x), \nu^{(1)}_1(x), \ldots, \nu^{(l-1)}_{l-1}(x),
\]
as just constructed, have degrees
\[d_0 \leqslant d - 0, \; d_1 \leqslant d_1 - 1, \ldots, d_{l-1} \leqslant d - (l - 1).
\]
Furthermore, the Wronskian determinant \( \langle u_0, u_1^{(l)}, \ldots, u_{l-1}^{(l-1)} \rangle \) is a sum of \( l! \) terms of the form

\[
\frac{d^{i_0} u_0 (x)}{dx^{i_0}} \frac{d^{i_1} u_1^{(l)} (x)}{dx^{i_1}} \ldots \frac{d^{i_{l-1}} u_{l-1}^{(l-1)} (x)}{dx^{i_{l-1}}},
\]

where \( i_0, i_1, \ldots, i_{l-1} \) run over all permutations of \( 0, 1, \ldots, l - 1 \). Each such term is of degree

\[
\sum_{\lambda = 0}^{l-1} (d_{\lambda} - i_{\lambda}) = \sum_{\lambda = 0}^{l-1} \{d_{\lambda} - (l - \lambda - 1)\} \leq \sum_{\lambda = 0}^{l-1} \{(d - \lambda) - (l - \lambda - 1)\} = l (d - l + 1),
\]

whence the assertion.

[4] If \( P(x, y) \) is any polynomial in \( K[x, y] \), then write

\[
P_{ij} (x, y) = \frac{\partial^{i+j} P(x, y)}{i! j! \partial x^i \partial y^j} \quad (i, j = 0, 1, 2, \ldots).
\]

We denote by \( r \) and \( s \) two positive integers which will be fixed in the next section, by \( \xi \) and \( \eta \) two elements of \( K \), and by \( \vartheta \) a non-negative real number. We then say that \( P(x, y) \) is at least of index \( \vartheta \) at \( (\xi, \eta) \) if

\[
P_{ij} (\xi, \eta) = 0 \text{ for } i \geq 0, j \geq 0, \frac{i}{r} + \frac{j}{s} < \vartheta;
\]

in the special case \( \vartheta = 0 \), there are no conditions.

This definition can be replaced by an equivalent one, as follows. Denote by \( z \) an indeterminate. Then

\[
P(\xi + xz^s, \eta + yz^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{ij} (\xi, \eta) x^i y^j z^{rs \left( \frac{i}{r} + \frac{j}{s} \right)}, = P \langle z \rangle \text{ say},
\]

becomes a polynomial in \( z \) with coefficients in \( K[x, y] \). This formula shows that \( P(x, y) \) is at least of index \( \vartheta \) at \( (\xi, \eta) \) if, and only if, \( P \langle z \rangle \) is divisible by \( z^{rs \vartheta} \) (i.e. all powers of \( z \) occurring in \( P \langle z \rangle \) must have exponents not less than \( rs \vartheta \)). If we multiply several such expressions

\[
P_0 \langle z \rangle, P_1 \langle z \rangle, \ldots, P_{l-1} \langle z \rangle
\]

which are divisible by

\[
z^{rs \vartheta_0}, z^{rs \vartheta_1}, \ldots, z^{rs \vartheta_{l-1}},
\]

respectively, then the product is divisible by

\[
z^{rs (\vartheta_0 + \vartheta_1 + \ldots + \vartheta_{l-1})},
\]

Therefore the following result holds:

**Lemma 3:** If, for \( \lambda = 0, 1, \ldots, l - 1 \), the polynomial \( P_\lambda(x, y) \) in \( K[x, y] \) is at least of index \( \vartheta_\lambda \) at \( (\xi, \eta) \), then

\[
P_0 (x, y) \ P_1 (x, y) \ldots P_{l-1} (x, y)
\]
is at least of index
\[ \partial_0 + \partial_1 + \ldots + \partial_{l-1} \]
at \((\xi, \eta)\).

[5] From now on,
\[ R(x, y) = \sum_{h=0}^{r} \sum_{k=0}^{s} R_{hk} x^h y^k \neq 0 \]
is a fixed polynomial in \(K[x, y]\) of degrees not greater than \(r\) in \(x\) and \(s\) in \(y\); here \(r\) and \(s\) are given positive integers. We further denote by
\[ \theta_0, \theta_1, \ldots, \theta_n \quad (n \geq 0) \]
a finite number of real numbers satisfying
\[ 0 < \theta_f \leq 1 \quad (f = 0, 1, \ldots, n), \]
and by
\[ \xi_0, \xi_1, \ldots, \xi_n \quad \text{and} \quad \eta_0, \eta_1, \ldots, \eta_n \]
two sets, each of \(n + 1\) elements of \(K\), such that no two elements of the
same set are equal.

Throughout this note, we make the assumption that \(R(x, y)\) is, for
\(f = 0, 1, \ldots, n\), at least of index \(\theta_f\) at \((\xi_f, \eta_f)\), so that
\[ R_{ij}(\xi_f, \eta_f) = 0 \quad \text{if} \quad i \geq 0, j \geq 0, \frac{i}{r} + \frac{j}{s} < \theta_f, f = 0, 1, \ldots, n. \]

[6] Since
\[ R(x, y) = \sum_{k=0}^{s} \left( \sum_{h=0}^{r} R_{hk} x^h \right) y^k, \]
the polynomial can be written in the form
\[ R(x, y) = \sum_{\lambda=0}^{l-1} u_{\lambda}(x) v_{\lambda}(y), \]
where the \(u\)'s are elements of \(K[x]\) of degrees not greater than \(r\), the \(v\)'s are polynomials in \(K[y]\) of degrees not greater than \(s\), and where
\[ 1 \leq l \leq \min (r, s) + 1. \]

Amongst all representations of this form, select one for which the number
\(l\) of terms is a minimum. Then both the \(l\) polynomials
\[ u_0(x), u_1(x), \ldots, u_{l-1}(x), \]
and the \(l\) polynomials
\[ v_0(y), v_1(y), \ldots, v_{l-1}(y), \]
are independent. For if, say, the $u$'s are not independent, then we may assume that $u_{l-1}(x)$ can be written as
\[ u_{l-1}(x) = \sum_{\lambda=0}^{l-2} a_{\lambda} u_{\lambda}(x) \]
where the coefficients $a_{\lambda}$ lie in $K$; therefore
\[ R(x, y) = \sum_{\lambda=0}^{l-2} u_{\lambda}(x) \{ v_{\lambda}(y) + a_{\lambda} v_{l-1}(y) \} \]
becomes a sum of only $l - 1$ terms, contrary to the definition of $l$.

We conclude therefore from Lemma 1 that neither of the two Wronskian determinants
\[ U(x) = \langle u_0(x), u_1(x), \ldots, u_{l-1} (x) \rangle \quad \text{and} \quad V(y) = \langle v_0(y), v_1(y), \ldots, v_{l-1}(y) \rangle \]
vanishes identically. Moreover, by Lemma 2,
\[ U(x) \text{ is at most of degree } l(r-l+1) \text{ in } x, \]
and
\[ V(y) \text{ is at most of degree } l(s-l+1) \text{ in } y. \]

[7] Denote by $(x - \xi_f)^{rf}$, where $f = 0, 1, \ldots, n$, the highest power of $x - \xi_f$ dividing $U(x)$, and by $(y - \eta_f)^{sf}$, where $f = 0, 1, \ldots, n$, the highest power of $y - \eta_f$ dividing $V(y)$. Since all the $\xi$'s and also all the $\eta$'s are different, $U(x)$ is divisible by
\[ \prod_{f=0}^{n} (x - \xi_f)^{rf}, \]
and $V(y)$ is divisible by
\[ \prod_{f=0}^{n} (y - \eta_f)^{sf}. \]
Therefore, on comparing the degrees, we obtain the two inequalities,
\[ r_0 + r_1 + \ldots + r_n \leq l(r - l + 1), \]
\[ s_0 + s_1 + \ldots + s_n \leq l(s - l + 1). \]

[8] We next introduce the determinant
\[ W(x, y) = |R_{x, y}(x, y)|_{x, y=0,1,\ldots,l-1}. \]
Since
\[ R_{x, y}(x, y) = \frac{1}{x! \mu!} \sum_{\lambda=0}^{l-1} \frac{u^{(\lambda)}(x)}{\lambda!} v^{(\mu)}(y), \]
the product rule of determinants leads to the identity,
\[ U(x) V(y) = \frac{1!2! \ldots (l-1)!}{2} W(x, y), \]
so that also $W(x, y)$ does not vanish identically.
Let $f$ be one of the indices $0, 1, \ldots, n$. Then, by hypothesis, $R(x, y)$ is at least of index $\theta_f$ at $(\xi_f, \eta_f)$; therefore $R_{x,y}(x, y)$ is at least of index

$$\max \left( 0, \theta_f - \frac{\lambda}{r} - \frac{\mu}{s} \right)$$

at $(\xi_f, \eta_f)$.

Now $W(x, y)$ is a sum of $l!$ terms of the form

$$\pm R_{i_0,0}(x, y) R_{i_1,1}(x, y) \ldots R_{i_{l-1},l-1}(x, y),$$

where $i_0, i_1, \ldots, i_{l-1}$ run over all permutations of $0, 1, \ldots, l-1$. By Lemma 3, such a term is at least of index

$$\sum_{\lambda=0}^{l-1} \max \left( 0, \theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s} \right) \geq \sum_{\lambda=0}^{l-1} \max \left( -\frac{i_\lambda}{r}, \theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s} \right) = \sum_{\lambda=0}^{l-1} \frac{i_\lambda}{r},$$

at $(\xi_f, \eta_f)$. Since

$$\sum_{\lambda=0}^{l-1} \frac{i_\lambda}{r} = \sum_{\lambda=0}^{l-1} \frac{\lambda}{r} = \frac{l(l-1)}{2r},$$

the whole determinant $W(x, y)$ is therefore also at least of index

$$\sum_{\lambda=0}^{l-1} \max \left( 0, \theta_f - \frac{\lambda}{s} \right) - \frac{l(l-1)}{2r}$$

at $(\xi_f, \eta_f)$.

On the other hand, $U(x)V(y)$ is divisible exactly by

$$(x-\xi_f)^{r_f} (y-\eta_f)^{s_f},$$

so that

$$\frac{\partial^{i+j}}{i! j!} \frac{\partial x^i \partial y^j}{\partial x^i \partial y^j} \frac{U(x)V(y)}{x=\xi_f, y=\eta_f} = 0 \text{ if } i \geq 0, j \geq 0, \frac{i}{r} + \frac{j}{s} < \frac{r_f}{r} + \frac{s_f}{s},$$

$$\neq 0 \text{ if } i = r_f, j = s_f.$$

From the identity

$$U(x)V(y) = \frac{1! 2! \ldots (l-1)!}{l!} W(x, y),$$

we therefore deduce the relations

$$\sum_{\lambda=0}^{l-1} \max \left( 0, \theta_f - \frac{\lambda}{s} \right) - \frac{l(l-1)}{2r} \leq \frac{r_f}{r} + \frac{s_f}{s} \quad (f = 0, 1, \ldots, n).$$
On adding these $n + 1$ inequalities and the two inequalities (I), we obtain the final inequality

$$\sum_{f=0}^{n} \sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) \leq (n + 1) \frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s}, \quad (II)$$

where now the unknown degrees $r_f$ and $s_f$ no longer occur.

\[ [11] \] The double sum on the left-hand side of (II) is easily replaced by a simple one. Put

$$A_f = \min\left(\lfloor \theta_f s \rfloor + 1, l\right) \quad (f = 0, 1, \ldots, n),$$

so that

$$\max\left(0, \theta_f - \frac{\lambda}{s}\right) = \begin{cases} \theta_f - \frac{\lambda}{s} & \text{if } 0 \leq \lambda \leq A_f - 1, \\ 0 & \text{if } \lambda \geq A_f. \end{cases}$$

Therefore

$$\sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) = \sum_{\lambda=0}^{A_f-1} \left(\theta_f - \frac{\lambda}{s}\right) = \frac{1}{2} A_f \left(2 \theta_f - \frac{A_f - 1}{s}\right),$$

so that the left-hand side of (II) may be written as

$$\frac{1}{2} \sum_{f=0}^{n} A_f \left(2 \theta_f - \frac{A_f - 1}{s}\right).$$

In order to simplify further, put

$$X = \frac{l}{s}, \quad X_f = \min\left(\theta_f, X\right) \quad (f = 0, 1, \ldots, n).$$

Then

$$sX_f = \min\left(s \theta_f, sX\right) = \min\left(s \theta_f, l\right)$$

and

$$A_f - 1 \leq sX_f \leq A_f, \text{ hence } A_f \left(2 \theta_f - \frac{A_f - 1}{s}\right) \geq sX_f \left(2 \theta_f - X_f\right).$$

Therefore (II) implies that

$$\frac{s}{2} \sum_{f=0}^{n} X_f \left(2 \theta_f - X_f\right) \leq (n + 1) \frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s}.$$

Next, the right-hand side of this inequality may be written as

$$+ \left(\frac{l}{s} + (n-1) \frac{l(l-1)}{2r}\right) = s(2X - X^2) \left(1 + \frac{1}{2-X} \left(\frac{1}{s} + \frac{(n-1)(l-1)}{2r}\right)\right).$$
Because, by \[6\],
\[ l \leq \min (r, s) + 1 \leq s + 1, \]
the inequality becomes therefore
\[
\sum_{f=0}^{n} X_f (2 \theta_f - X_f) \leq 2 \left\{ 1 - (1 - X)^2 \right\} \left( 1 + \frac{1}{2 - X} \left( \frac{1}{s} + \frac{(n-1)}{2r} \right) \right);
\]

[12] So far, \( r \) and \( s \) have been left arbitrary. Let now \( \delta \) be a number satisfying
\[ 0 < \delta \leq 1, \]
and restrict \( r \) and \( s \) by the conditions,
\[ s \geq \frac{5}{\delta} \geq 5, \quad r \geq \frac{5(n-1)}{2\delta}. \]

Then
\[ X = \frac{1}{s} \leq \frac{s+1}{s} \leq 1 + \frac{1}{s}, \quad 2 - X \geq \frac{0}{\delta}, \quad \frac{1}{s} \leq \frac{\delta}{5}, \quad \frac{5}{2r} \leq \frac{\delta}{5}, \]
so that
\[ \frac{1}{2 - X} \left( \frac{1}{s} + \frac{(n-1)}{2r} \right) \leq \frac{5}{4} \left( \frac{\delta}{5} + \frac{\delta}{5} \right) = \frac{\delta}{2}, \]
and our inequality takes the simple form
\[ \sum_{f=0}^{n} X_f (2 \theta_f - X_f) \leq (2 + \delta) \left\{ 1 - (1 - X)^2 \right\}. \]

But, for \( f = 0, 1, \ldots, n, \)
\[ X_f (2 \theta_f - X_f) - \theta_f^2 \left\{ 1 - (1 - X)^2 \right\} \equiv \theta_f^2 (1 - X)^2 - (\theta_f - X_f)^2 \]
is not negative, since either \( X \geq \theta_f, \) when \( X_f = \theta_f \) and
\[ \theta_f^2 (1 - X)^2 - (\theta_f - X_f)^2 = \theta_f^2 (1 - X)^2 \geq 0; \]
or \( X < \theta_f, \) when \( X_f = X \) and \( X \leq 1 \) and therefore
\[ \theta_f^2 (1 - X)^2 - (\theta_f - X_f)^2 \equiv X (1 - \theta_f) \left\{ \theta_f (1 - X) + (\theta_f - X) \right\} \geq 0. \]
Hence
\[ \left\{ 1 - (1 - X)^2 \right\} \sum_{f=0}^{n} \theta_f^2 \leq \sum_{f=0}^{n} X_f (2 \theta_f - X_f) \leq (2 + \delta) \left\{ 1 - (1 - X)^2 \right\}, \]
and since \( (1 - X)^2 \leq 1 \), we obtain finally the result,
\[ \sum_{f=0}^{n} \theta_f^2 \leq 2 + \delta. \]
Our discussion has thus led us to the following theorem:

**Theorem 1:** Let \( \delta, \theta_0, \theta_1, \ldots, \theta_n \) be \( n + 2 \) real numbers satisfying
\[
0 < \delta \leq 1, \ 0 < \theta_0 \leq 1, \ 0 < \theta_1 \leq 1, \ldots, \ 0 < \theta_n \leq 1,
\]
and let \( r \) and \( s \) be two integers satisfying
\[
s \geq \frac{5}{\delta}, \quad r \geq \frac{5(n-1)s}{2 \delta}.
\]

Let
\[
R(x, y) \neq 0
\]
be a polynomial of degrees not greater than \( r \) in \( x \) and \( s \) in \( y \), with coefficients in a field \( K \) of characteristic zero; write
\[
R_{ij}(x, y) = \frac{\partial^{i+j} R(x, y)}{i! j! \partial x^i \partial y^j} \quad (i, j = 0, 1, 2, \ldots).
\]

Further let
\[
\xi_0, \xi_1, \ldots, \xi_n \quad \text{and} \quad \eta_0, \eta_1, \ldots, \eta_n
\]
be two sets, each of \( n + 1 \) elements of \( K \), such that no two elements of the same set are equal. If now
\[
R_{ij}(\xi_f, \eta_f) = 0 \quad \text{for} \quad i \geq 0, \ j \geq 0, \ \frac{i}{r} + \frac{j}{s} < \theta_f, \quad f = 0, 1, \ldots, n,
\]
then
\[
\theta_0^2 + \theta_1^2 + \ldots + \theta_n^2 \leq 2 + \delta.
\]

In a second paper, I shall prove an analogous theorem for polynomials of the form
\[
\Sigma \Sigma R_{hk} x^h y^k \quad \left( h \geq 0, k \geq 0, \frac{h}{r} + \frac{k}{s} \leq 1 \right),
\]
and apply this result to the study of the continued fractions of algebraic numbers.

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