ON THE GENERATING FUNCTION OF THE INTEGERS WITH A MISSING DIGIT

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Let $n$ be a positive integer such that no digit in its decimal representation is equal to zero, and let $\mathcal{N}$ be the set of all such integers $n$. It is well known that the series

$$\sigma = \sum_{n \in \mathcal{N}} \frac{1}{n}$$

converges. Whether its value $\sigma$ is a transcendental number, or whether it can be expressed by means of elementary transcendental functions, is, however, a difficult question. In this note, I shall discuss the related series

$$f(z) = \sum_{n \in \mathcal{N}} z^n$$

with which $\sigma$ is connected by the relation

$$\sigma = \int_0^1 \frac{f(z)}{z} \, dz.$$

I shall prove that if $z$ is an algebraic number such that

$$0 < |z| < 1,$$

then $f(z)$ is a transcendental number; and a similar result holds for infinitely many similar functions.

1. The problem. Let $q \geq 2$ be a fixed positive integer. Every non-negative integer $n$ can be written in a unique way as a $q$-adic sum

$$n = h_0 + h_1 q + \ldots + h_r, \quad q^r = (h_0, h, \ldots, h_r),$$

where $h_0, h_1, \ldots, h_r$ are integers $0, 1, \ldots, q - 1$, and where, in particular, $h_r \neq 0$. For $n = 0$, we write $0 = (0)$. Let

This paper is the translation of one which appeared originally in the Chinese journal K’o Hsueh (Science), Vol. 29, (1947), p. 265-267, under the title 'Mou Chung T’e Pieh Cheng Shu Jih Ch’an Sheng Han Shu'.

Received January 3, 1951.
Let \( k \) be a fixed one of the integers 0, 1, ..., \( q-1 \), and let \( \mathcal{N}(k) \) be the set of all those integers \( n \geq 0 \) whose digits \( h_p \) are all different from \( k \),
\[
n = (h_0, h_1, ..., h_r) \geq 0, \ 0 \leq h_p \leq q-1, \ h_p \neq k \ (p = 0, 1, ..., r).
\]
We shall study here the properties of the generating function
\[
f_k(z) = \sum_{n \in \mathcal{N}(k)} z^n
\]
of \( \mathcal{N}(k) \).

2. The functional equation for \( f_k(z) \). It is clear that \( f_k(z) \) is majorized by the series \( 1 + z + z^2 + ... = (1 - z)^{-1} \) and so converges absolutely for \( |z| < 1 \).

There exists a functional equation between \( f_k(z) \) and \( f_k(z^q) \) which takes different forms for \( k = 0 \) and for \( k \neq 0 \).

I. \( k = 0 \). If \( n = (h_0, h_1, ..., h_r) \) belongs to \( \mathcal{N}(0) \), then the following two cases arise:

(i) \( r = 0 \), \( n = h_0 \), so that \( n \) is one of the integers \( 1, 2, ..., q-1 \).

(ii) \( r \geq 1 \), so that \( n \) can be written as \( n = h_0 + qn' \) where \( 1 \leq h_0 \leq q-1 \), \( n' = (h_1, h_2, ..., h_r) \in \mathcal{N}(0) \).

Therefore
\[
f_0(z) = \sum_{h_0=1}^{q-1} \left\{ z^{h_0} + \sum_{n' \in \mathcal{N}(0)} z^{h_0+qn'} \right\},
\]
so that
\[
f_0(z) = \frac{z-z^q}{1-z} (1+f_0(z^q)). \quad (I)
\]

II. \( k = 1, 2, ..., q-1 \). If \( n \) belongs to \( \mathcal{N}(k) \), then we can write
\[
n = (h_0, h_1, ..., h_r) = h_0 + qn'
\]
where \( h_0 \) is one of the integers 0, 1, 2, ..., \( k-1, k+1, ..., q-1 \), and where
\[
n' = (h_1, h_2, ..., h_r) \in \mathcal{N}(k).
\]
It is now clear that
\[ f_k(z) = \sum_{h_0=0}^{q-1} \sum_{n', e \in N(k)} z^{h_0 + qn'}, \]
whence
\[ f_k(z) = \left( \frac{1 - z^q}{1 - z} - z^k \right) f_k(z^q). \] (II)

The functional equations (I) and (II) may be combined into the one equation
\[ f_k(z) = \left( \frac{1 - z^q}{1 - z} - z^k \right) (\varepsilon_k + f_k(z^q)) \quad (k = 0, 1, \ldots, q-1), \] (I)
where \( \varepsilon_k = 1 \) if \( k = 0 \), and \( \varepsilon_k = 0 \) if \( k = 1, 2, \ldots, q-1 \).

In the simplest case \( q = 2 \), we have
\[ f_0(z) = \sum_{\nu=1}^{\infty} z^{2\nu-1}, \quad f_0(z) = z + zf_0(z^2), \]
\[ f_1(z) = 1, \quad f_1(z) = f_1(z^2). \]

3. The analytic behaviour of \( f_k(z) \). It is clear from the definition that
\[ f_0(z) = z + z^2 + \ldots + z^{q-1} + \ldots, \]
\[ f_k(z) = 1 + z + \ldots + z^{k-1} + z^{k+1} + \ldots \quad (k = 1, \ldots, q-1), \]
whence, for \( |z| < 1 \),
\[ \lim_{\nu \to \infty} f_k(z^{q^\nu}) = 1 - \varepsilon_k \quad (k = 0, 1, \ldots, q-1). \] (2)

We further deduce from the functional equations (I) and (II) that
\[ f_0(z) = \frac{z - z^q}{1 - z} + \frac{z - z^q}{1 - z} \frac{z^q - z^{q^2}}{1 - z^q} + \ldots \]
\[ + \frac{z - z^q}{1 - z} \frac{z^q - z^{q^2}}{1 - z^q} \ldots \frac{z^{q^{\nu-1}} - z^{q^\nu}}{1 - z^{q^{\nu-1}}} (1 + f_0(z^{q^\nu})), \] (3)
and
\[ f_k(z) = \left( \frac{1 - z^q}{1 - z} - z^k \right) \left( \frac{1 - z^{q^2}}{1 - z^q} - z^{kq} \right) \ldots \]
\[ \times \left( \frac{1 - z^{q^\nu}}{1 - z^{q^{\nu-1}}} - z^{kq^{\nu-1}} \right) f_k(z^{q^\nu}), \quad (k = 1, 2, \ldots, q-1). \] (4)
Theorem 1. If the special case \( q = 2, k = 1 \) is excluded, then \( f_k(z) \) is regular inside the unit circle and has this circle as its natural boundary.

Proof. Let \( \kappa \) and \( \lambda \) be two non-negative integers; put

\[
\theta = e^{\frac{2\pi i \kappa}{q^\lambda}}.
\]

Assume that \( \kappa \) is prime to \( q \) so that \( \theta \) is a primitive \( q^\lambda \)-th root of unity. It is obvious that for \( \lambda \geq 1 \) none of the polynomials

\[
\frac{z^{q^\nu - 1} - z^\nu}{1 - z^{q^\nu - 1}}, \quad \frac{1 - z^{q^\nu}}{1 - z^{q^\nu - 1} - z^{q^\nu - 1}} \quad (v = 1, 2, \ldots, \lambda)
\]

in \( z \) vanishes if \( z = \theta \). On the other hand, if the case \( q = 2, k = 1 \) is excluded, then evidently

\[
\lim_{r \to 1} f_k(r) = +\infty \quad (5)
\]

as \( r \) tends to 1 along the real interval \( 0 \leq r < 1 \). But then, by \( \theta^{q^\lambda} = 1 \), from (3), (4), and (5), also

\[
\lim_{r \to 1} f_k(\theta r) = \infty.
\]

Now the points \( \theta \) are everywhere dense on the unit circle, and the assertion follows at once.

Corollary. Except for the case \( q = 2, k = 1 \), \( f_k(z) \) is a transcendental function of \( z \).

4. The arithmetic behaviour of \( f_k(z) \). Some twenty years ago, I proved a result in which the following theorem is contained as a special case [Mathematische Annalen, 101 (1929), 332-366].

Theorem 2. Let \( q \geq 2 \) be a fixed integer, and let

\[
F(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^\nu
\]

be a power series with the following properties:
(i) All $a_v$ are rational numbers.
(ii) $F(z)$ converges in a neighbourhood of $z = 0$.
(iii) $F(z)$ is not an algebraic function of $z$.
(iv) $F(z)$ satisfies a functional equation of the form

$$F(z^q) = \frac{a(z)F(z) + b(z)}{c(z)F(z) + d(z)},$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ are polynomials with rational coefficients such that $\triangle(z) = a(z)d(z) - b(z)c(z)$ does not vanish identically in $z$. Then if $z$ is an algebraic number satisfying

$$0 < |z| < 1, \quad \triangle(z^q) \neq 0 \quad (v = 0, 1, 2, \ldots),$$

$F(z)$ is a transcendental number, but not a Liouville number.

If we apply this theorem to $F(z) = f_k(z)$, then

$$a(z) = 1, \quad b(z) = -\frac{z - z^q}{1 - z}, \quad c(z) = 0, \quad d(z) = \frac{z - z^q}{1 - z},$$
or

$$a(z) = 1, \quad b(z) = c(z) = 0, \quad d(z) = \frac{1 - z^q}{1 - z} - z^k,$$

according as to whether $k = 0$ or $1 \leq k \leq q - 1$. We therefore obtain the following result.

**Theorem 3.** Let the case $q = 2$, $k = 1$ be excluded. If $z$ is an algebraic number which satisfies the inequality

$$0 < |z| < 1 \quad \text{for } k = 0,$$

and the inequalities

$$0 < |z| < 1, \quad \frac{1 - z^q}{1 - z^q - 1} - z^{kq-1} = o(v = 1, 2, \ldots) \quad \text{for } 1 \leq k \leq q - 1,$$

then $f_k(z)$ is a transcendental number, but not a Liouville number. Furthermore

$$f_k(0) = 1 - \varepsilon_k \quad (k = 0, 1, \ldots, q - 1),$$

and if $k = 1, 2, \ldots, q - 1$, $0 < |z| < 1$ and there is a $v = 1, 2, \ldots$, such that
\[
\frac{1 - z^{q^\nu}}{1 - z^{q^\nu-1}} - z^{kq^\nu-1} = 0,
\]
then \( f_k(z) = 0. \)

5. **The zeros of \( f_k(z) \).** The polynomials

\[
\phi_k(z) = \frac{1 - z^d}{1 - z} - z^k \quad (k = 1, 2, \ldots, q-1)
\]
satisfy the functional equations

\[
\phi_k(1/z) = z^{-(q-1)}\phi_{q-k-1}(z). \tag{6}
\]

Let us assume that \( \phi_k(z) \) has \( \mu(k) \) zeros of absolute value less than 1, and \( \nu(k) \) zeros of absolute value equal to 1. From

\[
\begin{align*}
\phi_{q-1}(z) &= 1 + z + z^2 + \cdots + z^{q-2} \quad (q \text{ arbitrary}), \\
\phi_{(q-1)/2}(z) &= (1 + z + \cdots + z^{(q-3)/2})(1 + z^{(q+1)/2}) \quad (q \text{ odd}),
\end{align*}
\]

it is clear that

\[
\mu(k) = 0 \text{ if } k = q-1, \text{ or if } k = (q-1)/2.
\]

Further from (6),

\[
\nu(k) = \nu(q-k-1). \tag{7}
\]

**Theorem 4.** Let \( 1 \leq k \leq q-2 \) and \( k \neq (q-1)/2 \). Then \( \mu(k) > 0 \).

**Proof.** The polynomial \( \phi_k(z) \) is of exact degree \( q-1 \); it suffices therefore to prove that \( \nu(k) < q-1 \). For the product of the zeros of \( \phi_k(z) \) is evidently equal to \( 1 \): hence if at least one zero is of absolute value different from 1, then there is also at least one zero of absolute value less than 1.

Since \( k \neq (q-1)/2 \), it suffices to prove this inequality for \( \nu(k) \) if

\[
k = 1, 2, \ldots, [(q-2)/2].
\]

We first note that \( \phi'_k(z) \) has no multiple zeros on the unit circle. For at such zeros,
\[ 1 - z^q - z^k + z^{k+1} = 0, \quad qz^{q-1} + kz^{k-1} - (k+1)z^k = 0, \]

therefore

\[ (q-k)z^q = z^{k+1} - k, \]

whence, by \(|z| = 1,\)

\[ q-k \leq k+1, \quad k \geq (q-1)/2, \]

contrary to hypothesis.

Denote by

\[ \zeta = e^{\alpha i}, \] where \( 0 < \alpha < 2 \pi, \]
a zero, hence a simple zero, of

\[ \phi_k(z) = 1 + z + \ldots + z^{q-1} - z^k \]
on the unit circle. Since

\[ z^{-q-1/2} \phi_k(z) = \frac{z^q - z^{q/2} - \ldots - z^{q/2} - z^{q-2k-1/2}}{z^{q/2} - \ldots - z^{q/2} - z^{q-2k-1}}, \]

necessarily

\[ \frac{\sin q\alpha/2}{\sin \alpha/2} = \cos \frac{q - 2k - 1}{2} \alpha - i \sin \frac{q - 2k - 1}{2} \alpha, \]

and so

\[ \sin \frac{q - 2k - 1}{2} \alpha = 0. \]

Hence

\[ \alpha = \frac{2n\pi}{q - 2k - 1}, \]

where \( n \) is one of the integers \( 1, 2, \ldots, q - 2k - 1 < q - 1. \)

From this the assertion \( v(k) < q - 1 \) follows at once.

Let us combine the last results. We have found:
Theorem 5. If \( k = q - 1 \), or \( k = (q - 1)/2 \), then \( f_k(z) \) has no zeros inside the unit circle. If \( k = 0 \), then \( f_0(z) \) has the algebraic zero \( z = 0 \), and all its possible other zeros are transcendental. In all other cases, the zeros of \( f_k(z) \) are algebraic numbers, and there are an infinity of them inside the unit circle.

In a similar way, the generating function of integers with more than one missing digit, or with a missing sequence of digits can be investigated.

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