ON THE LATTICE DETERMINANTS OF TWO PARTICULAR
POINT SETS

K. MAHLER*.


It is well known that the star domain

\[ K : |xy| \leq 1 \]

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has the determinant \( \Delta(K) = \sqrt{5} \). Further, for the ray domain
\[
R_0: \quad 0 \leq xy \leq 1, \quad x \geq 0, \quad y \geq 0,
\]
forming the part of \( K \) in the first quadrant, B. Segre [4] and I [1] have shown that \( \Delta(R_0) = 1 \). In this note I determine the determinant of the ray domain
\[
R: \quad |xy| \leq 1, \quad x \geq 0,
\]
consisting of the intersection of \( K \) with the half-plane \( x \geq 0 \), and more generally that of its subset
\[
R_t: \quad |xy| \leq 1, \quad x \geq t,
\]
where \( t \) is an arbitrary non-negative constant. Since
\[
K \supset R \supset R_t,
\]
it is obvious that
\[
\Delta(R_t) \leq \Delta(R) \leq \sqrt{5}.
\]
Surprisingly enough, it can be proved that here the equality signs hold, so that
\[
\Delta(R_t) = \Delta(R) = \sqrt{5}.
\]

For let \( \Lambda \) be an arbitrary \( R_t \)-admissible lattice; it evidently suffices to show that \( d(\Lambda) \geq \sqrt{5} \). In every rectangle
\[
P_t: \quad |x| \leq t, \quad |y| \leq \tau,
\]
\( \Lambda \) has at most finitely many points. Hence a \( \tau \) with \( 0 < \tau \leq t^{-1} \) can be chosen such that \( O = (0, 0) \) is the only lattice point contained in \( P_t \). But then \( \Lambda \) is clearly an admissible lattice of the star domain
\[
K_t: \quad |xy| \leq 1, \quad |y| \leq \tau.
\]
Further \( \Delta(K_t) = \sqrt{5} \), as follows at once from Theorem 10 of my paper [2] on putting \( F(X) = \max (|xy|^4, \tau^{-1}|y|) \). Therefore
\[
\Delta(\Lambda) \geq \Delta(K_t) = \sqrt{5},
\]
as asserted.

Although the set \( R_t \) covers an arbitrarily small part of the halfplane \( x \geq 0 \), its determinant has just been shown to be positive and constant. I now give an example of a ray set, also of positive constant determinant, but filling an arbitrarily large portion of this halfplane \( x \geq 0 \).

Let now \( t > 0 \); denote by \( S_t \) the set of all points \((x, y)\) for which
\[
either 0 \leq x \leq t, or x > t and |xy| \leq 1.
\]

Then we shall prove that
\[
\Delta(S_t) = \frac{3 + \sqrt{5}}{2}.
\]
For let $\Lambda$ be any $S_t$-admissible lattice. This lattice contains points different from the origin in the parallel strip $|x| < t$, because this strip is convex, symmetric in the origin, and of infinite area. Since the lattice is $S_t$-admissible, such points necessarily lie on the $y$-axis. There exists then also a point $(0, a)$ of $\Lambda$ of smallest positive $a$, and this point is therefore primitive. Next we can select a second point $(b, c)$ of $\Lambda$ such that $(0, a)$ and $(b, c)$ together form a basis of $\Lambda$; the lattice consists thus of the points

$$(bv, au + cv) \quad (u, v = 0, \pm 1, \pm 2, \ldots).$$

There is no loss of generality in assuming that $b$ is also positive. Since $\Lambda$ is $S_t$-admissible, this means that

$$b \geq t$$

and that further

$$|bv(au + cv)| \geq 1 \quad \text{if} \quad u = 0, \pm 1, \pm 2, \ldots; \quad v = 1, 2, 3, \ldots.$$

Put

$$\xi = -\frac{c}{a}.$$

Since $d(\Lambda) = ab$, then

$$\left| \frac{u}{v} - \xi \right| \geq \frac{1}{d(\Lambda)v^2} \quad \text{if} \quad u = 0, \pm 1, \pm 2, \ldots; \quad v = 1, 2, 3, \ldots.$$

Now a theorem of A. V. Prasad [3] states that for every real $\xi$, integers $u, v \geq 1$ can always be chosen such that

$$\left| \frac{u}{v} - \xi \right| \leq \frac{2}{(3 + \sqrt{5})v^2}.$$ 

The last inequality implies then that

$$d(\Lambda) \geq \frac{3 + \sqrt{5}}{2},$$

and therefore

$$\Delta(S_t) \geq \frac{3 + \sqrt{5}}{2}.$$ 

Here the sign of equality holds. For select any $b \geq t$ and put

$$a = \frac{3 + \sqrt{5}}{2b}, \quad c = \frac{1 - \sqrt{5}}{2} a,$$

so that

$$d(\Lambda) = ab = \frac{3 + \sqrt{5}}{2}, \quad \xi = -\frac{c}{a} = \frac{\sqrt{5} - 1}{2}.$$ 

It is obvious that $\Lambda$ contains no points $(x, y)$ for which

$$0 < x < t.$$
Further, by Prasad's theorem, the inequality
\[ \frac{u}{v} - \frac{\sqrt{5} - 1}{2} \leq \frac{C}{v^2} \]
has no solutions in integers \( u, v \geq 1 \) if
\[ C < \frac{2}{3 + \sqrt{5}}. \]
This implies that there are no such integers for which
\[ |xy| = |bv(au + cv)| < 1, \]
and hence that \( \Lambda \) is \( S_1 \)-admissible. This concludes the proof.

References.

The University,
Manchester, 13.