ON THE COMPOSITION OF PSEUDO-VALUATIONS

BY

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To Prof. A. Ostrowski
on his 60th birthday

In this paper we consider pseudo-valuations \(^1\) on commutative rings \(R\) with a unit-element, and define certain processes for obtaining new pseudo-valuations from given ones or from real functions on \(R\). For each type of operation there are two definitions according to whether the resulting pseudo-valuation is, or is not, required to be non-archimedean. Given a real non-negative function \(\varphi\) on \(R\) we derive in Ch. I a pseudo-valuation which may be described as 'the greatest pseudo-valuation majorised by \(\varphi\)'. We then define the products and compounds of two pseudo-valuations and obtain some of their properties in Ch. II. These operations are illustrated in Ch. III by examples from algebraic number fields and rings of algebraic integers. In Ch. IV the connexion between the different operations, and their invariance properties, are established. The operations of forming the product and compound correspond to forming the sum and product of two ideals, just as the sum of pseudo-valuations corresponds to the intersection of ideals (P.I. 19). This suggests a number of relations connecting the different operations, which are in fact found to hold when we restrict ourselves to bounded non-archimedean pseudo-valuations. On the other hand, we can prove that certain laws such as the distributive law for multiplication (or compounding) and addition do not hold generally (Ch. V).

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\(^1\) Cf. the definition in § I, and K. Mahler, Über Pseudobewertungen I, Acta Mathematica 66 (1935) 79–119. This paper will be referred to as P. I, followed by the number of the paragraph.
I.

1. In all that follows $R$ is a commutative ring with the unit-element 1. A pseudo-valuation of $R$ is a real-valued function $W(a)$ defined for all $a$ in $R$ and having the properties

i) $W(0) = 0$, $W(a) > 0$;

ii) $W(a - b) < W(a) + W(b)$;

iii) $W(ab) < W(a) \cdot W(b)$.

We call a real-valued function on $R$ admissible, if it satisfies i), subadditive, if it satisfies ii) and submultiplicative, if it satisfies iii). A real-valued function $\varphi(a)$ which satisfies the inequality

$$\varphi(a - b) < \max \{\varphi(a), \varphi(b)\}$$

is called non-archimedean. Thus a function which satisfies i), ii) and iii) will be called a non-archimedean pseudo-valuation, which agrees with the usual terminology (as e.g. in P. I). For the sake of distinction the ordinary pseudo-valuations will sometimes be called subadditive.

As examples of pseudo-valuations we have the functions

$$U(a) = 0 \text{ and } W_0(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0. \end{cases}$$

They are called the improper pseudo-valuation and the trivial pseudo-valuation of $R$, respectively.

An admissible submultiplicative function $\varphi$ always satisfies

$$\varphi(1) > 1 \quad (1)$$

unless it is identically zero; for if $\varphi(a) \neq 0$ for some $a$ in $R$, then

$$0 < \varphi(a) = \varphi(1.a) < \varphi(1)\varphi(a),$$

from which (1) follows on dividing by $\varphi(a)$. Hence we have the lemma.

**Lemma 1.1.** An admissible submultiplicative function $\varphi$, which is such that $\varphi(1) < 1$, is identically zero.

In particular, a pseudo-valuation $W$ is improper if $W(1) < 1$.

2. In P.I. 7 it was shown that any finite set of pseudo-valuations $W_1, \ldots, W_n$ defines two new pseudo-valuations

$$W_X(a) = W_1(a) + \ldots + W_n(a)$$

and

$$W^*_X(a) = \max \{W_1(a), \ldots, W_n(a)\},$$

which are in fact equivalent 2).

If we have an infinite set of pseudo-valuations: $W_\lambda (\lambda \in \Lambda)$, then we can similarly define the functions

$$W_X(a) = \sum_{\lambda \in \Lambda} W_\lambda(a); \quad W^*_X(a) = \sup \{W_\lambda(a)\},$$

where the expressions on the right may assume the value $+\infty$. If $W_X$ or $W^*_X$ is finite for all $a$ in $R$, it will again define a pseudo-valuation; the proof is exactly as in the finite case. However, $W_X$ will not be finite unless almost all the $W_\lambda$ are improper. For every proper pseudo-valuation satisfies $W(1) > 1$, and so

$$W_X(1) > \sum_{\lambda \in \Lambda \text{ proper}} 1.$$  

Hence the sum $W_X$ gives nothing new, and we shall only consider the second type of sum.

We have the following obvious criterion for deciding when

$$\sup(W_\lambda(a))$$

is a pseudo-valuation.

**Lemma 2.1.** If $W_\lambda (\lambda \in \Lambda)$ is a non-empty family of pseudo-valuations on $R$ and if there is a real function $\varphi$ on $R$ such that

$$W_\lambda(a) < \varphi(a) \quad \text{for all } a \in R, \lambda \in \Lambda,$$  

then $\sup(W_\lambda(a))$ is a pseudo-valuation.

For the condition (2) ensures that $\sup(W_\lambda(a))$ shall be finite for all values of $a$.

3. The criterion of Lemma 2.1 suggests considering the pseudo-valuations which are majorised by a given real function $\varphi$ on $R$. Such pseudo-valuations exist if and only if $\varphi(a) > 0$ for all $a \in R$. Let $\Omega_R$ be the set of all pseudo-valuations on $R$. Then if $\varphi$ is any real non-negative function on $R$, we can put

$$W_\varphi(a) = \sup \{W(a) | W \in \Omega_R, W < \varphi\},$$

where $W < \varphi$ means $W(a) < \varphi(a)$ for all $a \in R$. By Lemma 2.1, $W_\varphi$ is a pseudo-valuation; in fact it is the greatest pseudo-valuation majorised by $\varphi$. Similarly we can define the greatest non-archimedean pseudo-valuation majorised by $\varphi$ as

$$W^*_\varphi(a) = \sup \{W(a) | W \in \Omega_R, W \text{ non-archimedean and } W < \varphi\}.$$

In particular, if we take as our non-negative function $\varphi$ a pseudo-valuation $V$, then $W_\varphi$ defined by (3) is the largest non-archimedean pseudo-valuation majorised by $V$. Of course this may be the improper pseudo-valuation $U'$; e.g. if $V$ is the ordinary absolute value on the field of rational numbers, then there is no proper non-

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2) Cf. P.I. 8, or § 14 below, for the definition of 'equivalent'.

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archimedean pseudo-valuation majorised by $V$, and so $W_P = U$ in this case. On the other hand, if $V$ is again taken to be the absolute value, this time on the ring of rational integers, then the largest non-archimedean pseudo-valuation majorised by $V$ is $W_0$, the trivial pseudo-valuation. This is not difficult to prove and also follows from the alternative definitions given later.

Let $\varphi$ be any real non-negative function on $R$. We can think of the set of pseudo-valuations $W$ majorised by $\varphi$ as determined by a set of conditions, one for each $a \in R$, viz. $W(a) \leq \varphi(a)$. Only the condition for $a = 0$ is vacuous, since $\varphi(0) \geq W(0) = 0$ holds in any case. Thus the value $\varphi(0)$ does not affect our set of pseudo-valuations and we may suppose from the outset that $\varphi(0) = 0$, i.e. that $\varphi$ is admissible. Then we can sum up the results of this paragraph in

**Theorem 3.1.** If $\varphi$ is any admissible function on $R$, then there is i) a uniquely determined greatest pseudo-valuation $W_\varphi$ which is majorised by $\varphi$ and ii) a uniquely determined greatest non-archimedean pseudo-valuation $W_P^\varphi$ which is majorised by $\varphi$.

Clearly

$$0 < W_P^\varphi < W_\varphi < \varphi.$$ Further, $W_\varphi$ coincides with $\varphi$ if and only if $\varphi$ is a pseudo-valuation, and $W_P^\varphi$ coincides with $\varphi$ if and only if $\varphi$ is a non-archimedean pseudo-valuation. It is also clear that $W_\varphi$ is the greatest non-archimedean pseudo-valuation majorised by $W_\varphi$.

4. The above definitions of $W_\varphi$ and $W_P^\varphi$ were non-constructive. We shall now give a constructive definition of these functions. For every admissible function $\varphi$ we define the function

$$\varphi^\times(a) = \inf_{\prod x_i \leq a} \prod \varphi(x_i),$$

where the greatest lower bound is extended over all factorisations of $a$ in $R$.

The function $\varphi^\times$ is again admissible. For the lower bound of a set of non-negative numbers exists and is non-negative, and since there are factorisations of $a$ (e.g. $a = a$), the set over which the lower bound is extended is not empty. Since $\varphi(0) = 0$, we have $\varphi^\times(0) = 0$ and so $\varphi^\times$ is admissible.

Further $\varphi^\times$ is submultiplicative. To prove this, let $a, b \in R$. Then for any factorisations $a = \prod x_i$, $b = \prod y_j$,

$$\varphi^\times(ab) \leq \prod \varphi(x_i) \prod \varphi(y_j).$$

Hence, by taking the lower bound over all such factorisations of $a$ and $b$, we get

$$\varphi^\times(ab) \leq \inf \prod \varphi(x_i) \inf \prod \varphi(y_j) = \varphi^\times(a) \varphi^\times(b)$$

as asserted.

The operation $\varphi \to \varphi^\times$ is monotone, i.e. if $\varphi_1 \leq \varphi_2$, then

$$\varphi_1^\times \leq \varphi_2^\times,$$

as is immediate from the definition.

If we take the factorisation $a = a$ in (5), we see that $\varphi^\times(a) \leq \varphi(a)$ for all $a \in R$. Thus $\varphi^\times \leq \varphi$. Equality holds here if (and of course only if) $\varphi$ is submultiplicative. For then

$$\varphi(a) \leq \prod \varphi(x_i)$$

for all factorisations of $a$, hence

$$\varphi(a) \leq \inf \prod \varphi(x_i) = \varphi^\times(a).$$

5. Next we define for every admissible function $\varphi$ a second function $\varphi^+$ by the rule

$$\varphi^+(a) = \inf_{\Sigma x_i = a} \max_i \varphi(\pm x_i),$$

where the lower bound is taken over all additive decompositions $\Sigma x_i = a$ and over all possible distributions of signs before the $x_i$.

As for $\varphi^\times$ we can verify that $\varphi^+$ is admissible. It is also non-archimedean, for if $\Sigma x_i = a$ and $\Sigma y_i = b$, then

$$\varphi^+(a - b) \leq \max \{\max_i \varphi(\pm x_i), \max_j \varphi(\pm y_j)\},$$

whence, on taking the lower bound,

$$\varphi^+(a - b) \leq \max \{\inf \max_i \varphi(\pm x_i), \inf \max_j \varphi(\pm y_j)\} = \max \{\varphi^+(a), \varphi^+(b)\}. $$

Similarly we define

$$\varphi^\oplus(a) = \inf_{\Sigma x_i = a} (\Sigma \varphi(\pm x_i))$$

and show that $\varphi^\oplus$ is admissible and subadditive.
We note that the operations $\varphi \to \varphi^+$ and $\varphi \to \varphi^\otimes$ are both monotone. Further $\varphi^+ < \varphi^\otimes < \varphi$, and $\varphi = \varphi^+$ or $= \varphi^\otimes$ if and only if $\varphi$ is non-archimedean or subadditive, respectively. This follows in the same way as for the function $\varphi^\times$.

**Lemma 5.1.** If $\varphi$ is an admissible submultiplicative function, then $\varphi^+$ and $\varphi^\otimes$ are likewise submultiplicative.

Proof. If $\Sigma x_i = a$, $\Sigma y_i = b$ are additive decompositions of $a$ and $b$ respectively, then $\Sigma x_i y_i = ab$. Hence, by the definition of $\varphi^+$, and because $\varphi$ is submultiplicative,

$$\varphi^+(ab) < \max_i \varphi(\pm x_i y_i)$$

$$< \max_i \varphi(\pm x_i) \varphi(\pm y_i)$$

$$= (\max_i \varphi(\pm x_i)) (\max_i \varphi(\pm y_i)).$$

Taking the lower bound over all decompositions of $a$ and $b$ we obtain

$$\varphi^+(ab) < \varphi^+(a) \varphi^+(b),$$

which shows that $\varphi^+$ is submultiplicative. The proof for $\varphi^\otimes$ is analogous.

We note that the roles of $+$ and $\times$ cannot be interchanged in this lemma, i.e. if $\varphi$ is an admissible non-archimedean (or subadditive) function, it does not follow that $\varphi^\times$ is again non-archimedean (or subadditive). In fact the proof of the lemma depends essentially on the distributive law, and this does not remain true if we interchange $+$ and $\times$.

6. If $\varphi$ is any admissible function, $\varphi^\times$ is both admissible and submultiplicative. Both these properties are preserved in $\varphi^\times \otimes$ which, moreover, is subadditive and hence is a pseudo-valuation. Similarly $\varphi^{\times+}$ is a non-archimedean pseudo-valuation.

**Theorem 6.1.** The function $\varphi^{\times \otimes}$ coincides with $W_\varphi$, the greatest pseudo-valuation majorised by $\varphi$. Similarly $\varphi^{\times+}$ equals $W_\varphi^{-}$, the greatest non-archimedean pseudo-valuation majorised by $\varphi$.

Proof. We have seen that $\varphi^\times \otimes$ is a pseudo-valuation. Since $\varphi^{\times \otimes} < \varphi^{\times+}$, it is majorised by $\varphi$; hence $\varphi^{\times \otimes} < W_\varphi$, because $W_\varphi$ is the greatest pseudo-valuation majorised by $\varphi$. Conversely, since $W_\varphi$ is a pseudo-valuation and $W_\varphi \leq \varphi$,

$$W_\varphi = W_\varphi^{\times \otimes} < \varphi^{\times \otimes},$$

whence $W_\varphi = \varphi^{\times \otimes}$. A similar argument shows that $W_\varphi = \varphi^{\times+}$.

We note that $\varphi^{\times+}$ and $\varphi^{\times \otimes}$ may be defined directly by

$$\varphi^{\times+}(a) = \inf \max_i \Pi_i \varphi(\pm x_i),$$

$$\varphi^{\times \otimes}(a) = \inf \Sigma_i \Pi_i \varphi(\pm x_i),$$

where in each case the lower bound is taken over all decompositions $a = \Sigma_i \Pi_i x_i$ and all distributions of signs.

II.

7. We now turn to the study of certain binary operations on $\Omega_R$, the set of pseudo-valuations on $R$. Their interpretation in the ring $R$ will be considered in Ch. IV, with the help of the results of Ch. I.

Let $a$ be any element of $R$. Since $a = a.1$, there always exist decompositions

$$a = x_1 y_1 + \ldots + x_n y_n \quad (1)$$

of $a$, where $x_i, y_i \in R$, and $n$ is arbitrary. Hence, for any two pseudo-valuations $W_1, W_2$ of $R$, we can form the lower bounds

$$W_1(a) = W_1 \circ W_2(a) = \inf \Sigma_i W_1(x_i) W_2(y_i), \quad (2)_1$$

$$W_{1\otimes}(a) = W_1 \otimes W_2(a) = \inf \Sigma_i W_1(x_i) W_2(y_i), \quad (2)_2$$

$$W_{1\times}(a) = W_1 \times W_2(a) = \inf \max_i (W_1(x_i), W_2(y_i)), \quad (2)_3$$

$$W_{1\otimes}(a) = W_1 \otimes W_2(a) = \inf \Sigma_i (W_1(x_i) + W_2(y_i)), \quad (2)_4$$

extended over all decompositions (1) of $a$. As we now show, the functions $W_{1\otimes} - W_{1\times}$ thus defined are pseudo-valuations on $R$.

8. As in the case of $\varphi^\times$ (§ 4) we can prove that the functions $W_N$, where $N = I, II, III, IV$, exist and are admissible. To prove that they are subadditive, let $b$ be a second element of $R$, and let $\varepsilon$ be any positive number. In each of the four cases $N = I, II, III, IV$ we select some decomposition (1) of $a$ and some decomposition

$$b = \xi_1 \eta_1 + \ldots + \xi_m \eta_m \quad (3)$$

of $b$ such that

$$W_1(a) > \max_i (W_1(x_i) W_2(y_i)) - \varepsilon \quad (4)_1$$

and

$$W_1(b) > \max_i (W_1(\xi_i) W_2(\eta_i)) - \varepsilon, \quad (4)_2$$

or
and
\[ W_{II}(a) > \sum_i (W_1(x_i)W_2(y_i)) - \varepsilon \]

or
\[ W_{III}(a) > \max_i(W_1(x_i), W_2(y_i)) - \varepsilon \]

and
\[ W_{IV}(a) > \sum_i (W_1(x_i) + W_2(y_i)) - \varepsilon \]

respectively.

By (1) and (3), \( a - b \) admits the decomposition
\[ a - b = x_1y_1 + \ldots + x_my_m + (-\xi_1)\eta_1 + \ldots + (-\xi_m)\eta_m; \]

further \( W_i(-\xi_i) = W_i(\xi_i) \). Hence, from the definition of \( W_N(a-b) \) and from the inequalities (4) for \( W_N(a) \) and \( W_N(b) \),
\[
W_I(a - b) < \max \{ \max_i(W_1(x_i)W_2(y_i)), \max_i(W_1(\xi_i)W_2(\eta_i)) \} \\
< \max(W_I(a), W_I(b)) + \varepsilon,
\]
\[
W_{II}(a - b) < \sum_i (W_1(x_i)W_2(y_i)) + \sum_i (W_1(\xi_i)W_2(\eta_i)) \\
< W_{II}(a) + W_{II}(b) + 2\varepsilon,
\]
\[
W_{III}(a - b) < \max \{ \max_i(W_1(x_i), W_2(y_i)), \max_i(W_1(\xi_i), W_2(\eta_i)) \} \\
< \max(W_{III}(a), W_{III}(b)) + \varepsilon,
\]
\[
W_{IV}(a - b) < \sum_i (W_1(x_i) + W_2(y_i)) + \sum_i (W_1(\xi_i) + W_2(\eta_i)) \\
< W_{IV}(a) + W_{IV}(b) + 2\varepsilon.
\]

In the limit, as \( \varepsilon \to 0 \), we see that \( W_N \) is in fact submultiplicative. Thus the following result has been proved:

**Theorem 8.1.** If \( W_1 \) and \( W_2 \) are arbitrary pseudo-valuations on \( R \), then \( W_1 \times W_2 \) and \( W_1 \circ W_2 \) are likewise pseudo-valuations on \( R \). Moreover \( W_1 \times W_2 \) and \( W_1 \circ W_2 \) are non-archimedean.

The functions \( W_1 \times W_2 \) and \( W_1 \circ W_2 \) will be called the non-archimedean product and the subadditive product, and the functions \( W_1 \times W_2 \) and \( W_1 \circ W_2 \) will be called the non-archimedean compound and the subadditive compound, of \( W_1 \) and \( W_2 \) respectively.

In establishing Theorem 8.1 we have not used the full force of the hypothesis. Instead of taking \( W_1 \) and \( W_2 \) to be pseudo-valuations it is enough to assume that they are admissible and submultiplicative and satisfy \( W_k(-a) = W_k(a) \) \((k = 1, 2) \). The last condition can also be dropped provided we modify the definitions (2) by allowing \( -x_i \) and \( -y_i \) as arguments on the right and taking the lower bound over all distributions of signs.
9. Just as the letter \( N \) was used to denote the four indices \( I, II, III, IV \) in \( W_N(a) \), we let the symbol \( \ominus \) stand for any one of the four signs , , \( \circ \), \( \times \), or \( \otimes \). Then it follows from the definitions and the commutativity of \( R \) that all four operations are commutative:

\[
W_1 \ominus W_2 = W_2 \ominus W_1. \tag{5}
\]

Here (5) is an abbreviation for \( W_1 \ominus W_2(a) = W_2 \ominus W_1(a) \) identically in \( a \), and similarly in later cases.

Next we prove that the four operations are associative. For this purpose we make use of

**Lemma 9.1.** If \( W_1, W_2, W_3 \) are any three pseudo-valuations of \( R \) (or indeed any admissible functions), then the function

\[
(W_1 \ominus W_2) \ominus W_3 \tag{6}
\]

(where \( \ominus \) is , , \( \ominus \), \( \times \) or \( \otimes \)) is symmetric in \( W_1, W_2, W_3 \).

Assuming the truth of this lemma for the moment, we have by (5),

\[
(W_1 \ominus W_2) \ominus W_3 = (W_2 \ominus W_3) \ominus W_1 = W_1 \ominus (W_2 \ominus W_3),
\]

and hence

**Theorem 9.2.** The operations , , \( \ominus \), \( \times \) and \( \otimes \) are commutative and associative.

10. It remains to prove Lemma 9.1. We do this by writing \( (W_1 \ominus W_2) \ominus W_3(a) \) as the lower bound of expressions involving certain types of ternary decomposition of \( a \). For \( N = I, II, III, IV \) write

\[
W_N^+(a) = W_1 \ominus W_2(a),
\]

\[
V_N(a) = W_N^+ \ominus W_3(a) = (W_1 \ominus W_2) \ominus W_3(a).
\]

An upper bound for \( V_N(a) \) in terms of ternary decompositions is obtained as follows. Let

\[
a = x_1y_1z_1 + \ldots + x_ny_nz_n. \tag{6}
\]

Then

\[
V_I(a) < \max_i W_I^+(x_iy_i)W_3(z_i),
\]

\[
V_{II}(a) < \max_i W_{II}^+(x_iy_i)W_3(z_i),
\]

\[
V_{III}(a) < \max_i(W_{III}^+(x_iy_i), W_3(z_i)) < \max_i(W_1(x_i), W_2(y_i), W_3(z_i)).
\]

For the subadditive compound \( V_{IV} \) we need decompositions of the more general type

\[
a = \sum_{i=1}^{\infty} \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} x_{iqor}y_{iqr}z_{iqrs}. \tag{7}
\]

Then

\[
V_{IV}(a) < \sum_{i,q,r,s} \{ W_{IV}^+(x_{iqor}y_{iqr}z_{iqrs}) + W_3(z_{iqrs}) \}
\]

\[
< \sum_{i,q,r,s} \{ W_1(x_{iqo}) + W_2(y_{iqr}) + W_3(z_{iqrs}) \}. \tag{8}
\]

By taking the lower bounds of the right-hand sides for all ternary decompositions of the type (6) for \( N = I, II, III, \) and of the type (7) for \( N = IV \), we find that

\[
V_I(a) < \inf \max_i W_1(x_i)W_2(y_i)W_3(z_i),
\]

\[
V_{II}(a) < \inf \max_i W_1(x_i)W_2(y_i)W_3(z_i),
\]

\[
V_{III}(a) < \inf \max_i (W_1(x_i), W_2(y_i), W_3(z_i)),
\]

\[
V_{IV}(a) < \inf \sum_{i,q,r,s} (W_1(x_{iqo}) + W_2(y_{iqr}) + W_3(z_{iqrs})).
\]

We now show that equality holds in each case. The Lemma will follow from this, since the right-hand sides are obviously symmetric in the \( W_i \)’s.

I. The non-archimedean product.

If \( \varepsilon > 0 \), there exists a decomposition

\[
a = u_1z_1 + \ldots + u_nz_n \tag{9}
\]

of \( a \) such that

\[
W_I^+(a) > \max W_I^+(u_i)W_3(z_i) - \varepsilon. \tag{10}
\]

Write

\[
\omega = \max (W_3(z_i), 1),
\]

so that \( \omega \) is a finite constant not less than 1, which only depends on the decomposition (9) of \( a \).

Next choose decompositions

\[
u_i = x_{i1}y_{i1} + \ldots x_{in}y_{in} \tag{11}
\]

of \( u_1, \ldots, u_n \) such that

\[
W_I^+(u_i) = W_1, W_2(u_i) > \max_{1 \leq q \leq r} W_1(x_{iqo})W_3(y_{iqr}) - \frac{\varepsilon}{\omega} \tag{12}
\]

\( (i = 1, \ldots, n) \).
In general the numbers \( v_i \) will depend on \( i \), but by adding, if necessary, a number of zero terms 0.0 to the decomposition (11) of \( u_i \) we can arrange that the decompositions (11) all have the same number of terms, so that \( v_1 = v_2 = \ldots = v_n = v \) say.

In order to unify the notation we put \( z_{i\bar{q}} = z_i \) (\( \bar{q} = 1, \ldots, v; \ i = 1, \ldots, n \)). Then by (9) and (11),

\[
\begin{align*}
    a &= \sum_{i=1}^{n} z_i \sum_{\bar{q}=1}^{v} x_{i\bar{q}} y_{i\bar{q}} \\
    &= \sum_{\bar{q}=1}^{v} \sum_{i=1}^{n} x_{i\bar{q}} y_{i\bar{q}} z_{i\bar{q}}.
\end{align*}
\]

This is a ternary decomposition of the form (6), as becomes clear when we replace the index-pair \( i\bar{q} \) by a simple index \( \mu \) running from 1 to \( m = n v \).

By (10) and (12)

\[
V_{I}(a) > \max_{\mu} \max_{\bar{q}} \left\{ W_{1}(x_{i\bar{q}}) W_{3}(y_{i\bar{q}}) - \frac{\epsilon}{\omega} \right\} W_{3}(z_{i\bar{q}}) - \epsilon,
\]

and from the definition of \( \omega \)

\[
\frac{\epsilon}{\omega} W_{3}(z_{i\bar{q}}) < \epsilon.
\]

Hence

\[
V_{I}(a) > \max_{\mu} \max_{\bar{q}} \left\{ W_{1}(x_{i\bar{q}}) W_{3}(y_{i\bar{q}}) W_{3}(z_{i\bar{q}}) \right\} - 2\epsilon,
\]

or, on changing to the single index \( \mu \),

\[
V_{I}(a) > \max_{\mu} \left\{ W_{1}(x_{\mu}) W_{3}(y_{\mu}) W_{3}(z_{\mu}) \right\} - 2\epsilon.
\]

Letting \( \epsilon \to 0 \) and combining the result with (8), we find that

\[
(W_{1}, W_{3}) \cdot W_{3}(a) = \inf \max_{\mu} W_{1}(x_{\mu}) W_{3}(y_{\mu}) W_{3}(z_{\mu}), \tag{13}
\]

where the lower bound is extended over all decompositions (6) of \( a \). This proves the assertion for the case of the non-archimedean product.

The proofs in the other three cases are similar and we can therefore be more concise. In each case \( \epsilon \) is a fixed but arbitrary positive number.

II. The subadditive product.

We choose a decomposition (9) of \( a \) such that

\[
W_{II}^{*} \circ W_{3}(a) > \sum_{i} W_{II}^{*}(u_i) W_{3}(z_i) - \epsilon.
\]

Next put

\[
\omega = \sum_{i} W_{3}(z_i) + 1,
\]

and choose decompositions (11) of \( u_1, \ldots, u_n \) such that

\[
W_{II}^{*}(u_i) = W_{1} \circ W_{3}(u_i) > \sum_{i} W_{1}(x_{i\bar{q}}) W_{3}(y_{i\bar{q}}) - \frac{\epsilon}{\omega} (i = 1, \ldots, n).
\]

Then

\[
V_{II}(a) > \sum_{i} \left( \sum_{\bar{q}} W_{1}(x_{i\bar{q}}) W_{3}(y_{i\bar{q}}) - \frac{\epsilon}{\omega} \right) W_{3}(z_i) - \epsilon,
\]

and hence

\[
V_{II}(a) > \sum_{\bar{q}} W_{1}(x_{\bar{q}}) W_{3}(y_{\bar{q}}) W_{3}(z_{\bar{q}}) - 2\epsilon.
\]

Changing the notation as before, we obtain the formula

\[
(W_{1} \circ W_{3}) \circ W_{3}(a) = \inf \sum_{\mu} W_{1}(x_{\mu}) W_{3}(y_{\mu}) W_{3}(z_{\mu}),
\]

where the lower bound is again extended over all decompositions (6) of \( a \).

III. The non-archimedean compound.

We now choose the decomposition (9) of \( a \) so that

\[
W_{III}^{*} \times W_{3}(a) > \max_{i} (W_{III}^{*}(u_i), W_{3}(z_i)) - \epsilon,
\]

and decompositions (11) of \( u_1, \ldots, u_n \) such that

\[
W_{III}(u_i) = W_{1} \times W_{3}(u_i) > \max_{\bar{q}} (W_{1}(x_{i\bar{q}}), W_{3}(y_{i\bar{q}})) - \epsilon
\]

\((i = 1, \ldots, n)\).

Then

\[
V_{III}(a) > \max_{\bar{q}} \max_{i} (W_{1}(x_{i\bar{q}}), W_{2}(y_{i\bar{q}}), W_{3}(z_{i\bar{q}})) - \epsilon,
\]

whence

\[
V_{III}(a) > \max_{\bar{q}} \max_{i} (W_{1}(x_{i\bar{q}}), W_{2}(y_{i\bar{q}}), W_{3}(z_{i\bar{q}})) - 2\epsilon,
\]

and we obtain the result

\[
(W_{1} \times W_{3}) \times W_{3}(a) = \inf \max_{\mu} (W_{1}(x_{\mu}), W_{2}(y_{\mu}), W_{3}(z_{\mu})),
\]

where the lower bound is again taken over all decompositions (6) of \( a \).

IV. The subadditive compound.

Again we choose a decomposition (9) of \( a \) such that
\[ W_{IP}^* \otimes W_3(a) > \Sigma_i \left( W_{IP}^*(u_i) + W_3(x_i) \right) - \varepsilon, \]
and next decompositions (11) of \( u_1, \ldots, u_n \) such that
\[ W_{IP}^*(u_i) > \Sigma_q \left( W_3(x_{iq}) + W_3(y_{iq}) \right) - \frac{\varepsilon}{n}. \]
Then
\[ V_{IP}(a) > \Sigma_i \left\{ \sum_{q} \left( W_3(x_{iq}) + W_3(y_{iq}) \right) - \frac{\varepsilon}{n} + W_3(x_i) \right\} - \varepsilon = \Sigma_{iq} \left( W_3(x_{iq}) + W_2(y_{iq}) + W_3(x_{iq}) \right) - 2\varepsilon. \]

Now the decomposition
\[ a = \Sigma_{iq} x_{iq}y_{iq}z_{iq}, \]
is of the form (7) (the range of each of the suffixes \( \sigma \) and \( \tau \) is 1 for every \( i \) and they have therefore been omitted). Hence
\[ (W_1 \otimes W_2) \otimes W_3(a) = \inf \Sigma_{iqr} \left( W_3(x_{iqr}) + W_2(y_{iqr}) + W_3(z_{iqr}) \right), \]
where this time the lower bound is taken over all decompositions (7) of \( a \).

This completes the proof of Lemma 9.1.

From the proof of the Lemma it is clear why the different types of ternary decomposition (6) and (7) have to be considered. By renaming the suffixes we can regard any decomposition (7) of \( a \) as being of the form (6) 4), but then we must count certain factors repeatedly. For the products and the non-archimedean compound this is of no importance, and it is therefore sufficient to consider decompositions (6) when taking the lower bound. In the case of the subadditive compound we cannot make this simplification but have to consider all decompositions (7) when calculating the lower bound. Though we actually use only a decomposition of the form \( a = \Sigma x_{iq}y_{iq}z_{iq} \), it is not enough to take such decompositions, because they are not symmetric in the three sets of factors. The decomposition (7) is in fact the simplest type of ternary decomposition which suffices for our purpose. It is not the most general type, as we might have a decomposition
\[ a = \Sigma x_{iqr}y_{iqr}z_{iqr}, \]

4) This is precisely what we did when we replaced the suffix pair \( iq \) by \( \mu \) in order to obtain (13).


11. Before considering the products and compounds in greater detail we shall illustrate the definitions by a few examples.

First take \( R \) to be an algebraic number field \( K \) of finite degree \( n \) over the rational field \( F \). Suppose that of the \( n \) conjugate fields
\[ K^{(1)}, K^{(2)}, \ldots, K^{(n)} \]
of \( K \) the first \( r_1 \) are real and the remaining \( 2r_2 = n - r_1 \) fields
\[ K^{(r_1+1)}, K^{(r_1+2)}, K^{(r_1+3)}, \ldots, K^{(r_1+r_2)}, \ldots, K^{(r_1+r_2)}. \]
are complex conjugate in pairs. If \( a \) is any element of \( K \), denote by \( a^{(h)} \) the number conjugate to it in \( K^{(h)} \). Then \( a^{(h)} \) is real for \( h = 1, \ldots, r_1 \), while \( a^{(r_1+h)} \) and \( a^{(r_1+r_2+h)} \) are complex conjugate for \( h = 1, \ldots, r_2 \).

The field \( K \) has exactly \( r_1 + r_2 \) inequivalent absolute values, viz.
\[ \Omega^{(h)}(a) = |a^{(h)}| \quad (h = 1, \ldots, r_1 + r_2). \]

Further let \( p \) be an arbitrary prime ideal in the ring \( J \) of integers of \( K \). To \( p \) there corresponds a \( p \)-adic valuation of \( K \) (unique to within equivalence) which we denote by \( \Omega_p(a) \). This valuation is fully determined if we know that
\[ \Omega_p(a) = \varepsilon_p, \quad \text{where} \quad 0 < \varepsilon_p < 1, \]
for all elements \( a \) of \( K \) which have a denominator prime to \( p \) and a numerator divisible by the first and no higher power of \( p \).

It is known that every pseudo-value of \( K \) not equivalent to \( U \) or \( W_0 \) is equivalent to the sum of a finite number of valuations \( \Omega^{(h)} \) and \( \Omega_p \). The tables which follow give the results of the four operations \( W_1 \circ W_2 \) applied to the functions \( U, W_0, \Omega^{(h)} \) and \( \Omega_p \). In these tables \( k \) and \( h \) are distinct indices \( 1, 2, \ldots, r_1 + r_2 \), provided that \( r_1 + r_2 > 2 \); in the exceptional case \( r_1 + r_2 = 1 \) the row and column belonging to \( \Omega^{(h)} \) are to be omitted 5). Similarly \( p \) and \( q \) denote two distinct prime ideals of \( J \).


6) This is the case when \( K \) is the rational field or an imaginary quadratic field.
### Table 1

<table>
<thead>
<tr>
<th>$W_1 \circ W_2$</th>
<th>$U$</th>
<th>$W_0$</th>
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<th>$\Omega^{(j)}$</th>
<th>$\Omega_p$</th>
<th>$\Omega_q$</th>
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### Table 3

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<td>$V_q$</td>
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<td>$U$</td>
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<td>$U$</td>
<td>$V_q$</td>
</tr>
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</table>

The pseudo-valuations $V_p$ and $V_p^*$ occurring in the Tables 3 and 4 are defined by

$$V_p(a) = c_p^{(\nu)}$$

and

$$V_p^*(a) = c_p^{(\nu)} + c_p^{(-\nu)}$$

where $\nu$ is the integer determined by $\Omega_p(a) = c_p^{(\nu)}$. Clearly both $V_p$ and $V_p^*$ are equivalent to the valuation $\Omega_p$.

12. Most of the entries in these Tables are an immediate consequence of

**Lemma 12.1.** Let $R$ be a field, $W(a)$ a pseudo-valuation and $\Omega(a)$ a valuation of $R$. If there exists an element $a$ of $R$ such that

$$W(a) < 1 < \Omega(a),$$

then

$$W \cdot \Omega = W \circ \Omega = W \times \Omega = W \otimes \Omega = U.$$  

In the case of the products ($\cdot$ and $\circ$) only one inequality in (1) need be strict.

**Proof.** Since $\Omega(a) > 1$, $a$ is different from zero, and so the unit-element 1 admits the decompositions

$$1 = a^m \cdot a^{-m} \quad (m = 1, 2, \ldots).$$

By the hypothesis

$$0 < \lim_{m \to \infty} W(a^m) < \lim_{m \to \infty} W(a) = 0,$$

$$\lim_{m \to \infty} \Omega(a^{-m}) = \lim_{m \to \infty} \Omega(a)^{-m} = 0,$$

hence $W \circ \Omega \ (1) = 0$, and the Lemma follows by Lemma 1.1.
We shall discuss briefly the entries in the four Tables. Since the commutative law holds, we need only consider the entries on or above the main diagonal.

**Table 1.** We can use Lemma 12.1 for the non-diagonal elements, since all the argument functions except $U$ are valuations and all except $W_0$ are non-trivial.

$\Omega^A \Omega^A = U$. This follows by using the decomposition

$$1 = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1$$

of 1 and applying Lemma 1.1.

$\Omega_p \cdot \Omega_p = \Omega_p$. If

$$a = x_1 y_1 + \ldots + x_m y_m$$

is any decomposition of $a$, then

$$\Omega_p(a) = \Omega_p(\sum x_i y_i) < \max_i \{\Omega_p(x_i) \Omega_p(y_i)\},$$

hence $\Omega_p < \Omega_p \cdot \Omega_p$, and equality is established by using the decomposition $a = a \cdot 1$.

The same proof shows that $W_0 \cdot W_0 = W_0$, while it is evident that $U \cdot U = U$.

**Table 2.** As for Table 1, except that $\Omega^A \circ \Omega^A = \Omega^A$.

Clearly

$$\Omega^A(a) = \Omega^A(\sum x_i y_i) < \sum \Omega^A(x_i) \Omega^A(y_i),$$

for any decomposition (3) of $a$. Hence $\Omega^A < \Omega^A \circ \Omega^A$, and equality again follows by writing $a = a \cdot 1$.

**Table 3.** $W_0 \times W = W_0$, where $W$ is any pseudo-valuation occurring as argument in the Table. For if (3) is any decomposition of $a \not= 0$, then at least one $x_i$ is different from zero; therefore

$$\max_i \{|W_0(x_i), W(y_i)| > 1 = W_0(a),$$

and equality is attained for the decomposition $a = a \cdot 1$.

$U \times W = U$ ($W \not= W_0$). Let $a \not= 0$ be such that $W(a) < 1$, then

$$U \times W(a) < \max \{U(1/a), W(a)\} < 1,$$

hence $U \times W = U$.

The remaining non-diagonal elements follow from Lemma 12.1.

$\Omega^A \times \Omega^A = U$. As in Table 1, using the decomposition $1 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$ of 1.

$\Omega_p \times \Omega_p = V_p$. If (3) is any decomposition of $a \not= 0$, then

$$\max_i \{\Omega_p(x_i) \Omega_p(y_i)\} > \Omega_p(a).$$

Let $i = \mu$ be a suffix for which the maximum is attained, then

$$\Omega_p(x_\mu) \Omega_p(y_\mu) > \Omega_p(a).$$

Hence, by the definition of $V_p$,

$$\max \{\Omega_p(x_\mu), \Omega_p(y_\mu)\} > V_p(a),$$

and so $V_p < \Omega_p \times \Omega_p$. The reverse inequality follows by choosing $x$ and $y$ in $R$ such that both $xy = a$ and

$$\max \{\Omega_p(x), \Omega_p(y)\} = V_p(a).$$

**Table 4.** $U \otimes W = U$ ($W \not= W_0$). As for Table 3.

$W_0 \otimes W = W_0$ ($W \not= W_0$). Again, for $a \not= 0$,

$\sum W_0(x_i) + W(y_i) > 1 = W_0(a)$.

To obtain equality, take a decomposition $a = xy$ with $W(y) \to 0$. The proof that $W_0 \otimes W_0 = 2W_0$ is similar.

The remaining non-diagonal elements follow again from Lemma 12.1.

$\Omega^A \otimes \Omega^A = 2\Omega^A \otimes \Omega^A$. If (3) is any decomposition of $a$, then by the theorem of the arithmetic and geometric means and by Jensen's inequality,

$$\sum \{\Omega^A(x_i) + \Omega^A(y_i)\} > 2 \sum \{\Omega^A(x_i) \Omega^A(y_i)\}^{1/2}$$

$$> 2 \sum \{\Omega^A(x_i) \Omega^A(y_i)\}^{1/2}$$

$$> 2 \Omega^A(a)^{1/2}.$$

In this inequality the difference between the left- and righthand sides can be made arbitrarily small by choosing a decomposition $a = xy$ for which $|\Omega^A(x) - \Omega^A(y)|$ is sufficiently small.

$\Omega_p \otimes \Omega_p = V_p^*$. This follows as for the corresponding entry in Table 3.

13. As a second example consider the ring $J$ of all integers in the algebraic number field $K$ discussed in §§ 11—12. Let $\Omega^A$ and $\Omega_p$ have the same meaning as before; if $a$ is an ideal in $J$, denote by $W_0(a)$ the pseudo-valuation defined by

$$W_0(a) = \begin{cases} 0 \text{ if } a = 0 \pmod{a}, \\ 1 \text{ otherwise.} \end{cases}$$

It has been proved$^7$ that every pseudo-valuation of $J$ not equivalent to $U$ or $W_0$ is equivalent to the sum of a finite number of

absolute values $\Omega^{(a)}$, a finite number of $p$-adic valuations $\Omega_p$, and a residue-class pseudo-value $W_a$.

When $r_1 = r_2 = 1$, $J$ is the ring of rational integers or the ring of integers of an imaginary quadratic field; in these cases there is only one absolute value, which is moreover equivalent to $W_p$. We omit this case in the discussion which follows; thus in Tables 5—8 we suppose that $r_1 + r_2 > 2$. As before, $k_l$ and $k_2$ are two distinct integers 1, 2, $\ldots$, $r_1 + r_2$, and $p$ and $q$ are two distinct prime ideals of $J$. Further $a$ and $b$ are any two distinct ideals of $J$. Write $p^r \mid a$ to indicate that $a$ is divisible by $p^r$, but not by $p^{r+1}$. If $p^r \mid a$, the pseudo-value $V^{(a)}_p(a)$ is defined by

$$V^{(a)}_p(a) = \begin{cases} 0 & \text{if } a = 0 \pmod{p^r}, \\ \Omega_p(a) & \text{otherwise.} \end{cases}$$

Clearly $V^{(a)}_p$ is equivalent to $W_{p^r}$, unless $a = 0$.

We observe that $W_0$ and $U$ are the limiting cases of the residue-class pseudo-value $W_a$ with $a = 0$ or $J$ respectively. Therefore the proofs given for $W_a$ will hold for $W_0$ and $U$ if correctly interpreted.

**Table 5**

<table>
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<tr>
<th>$W_1$, $W_2$</th>
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<th>$W_0$</th>
<th>$\Omega^{(a)}$</th>
<th>$\Omega_p$</th>
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**Table 5.** It is clear that $W_a \cdot W_b = W_{a+b}$, and this holds even if $a = b$.

$\Omega^{(a)}, W = U$. Let $e$ be a unit of $J$ such that $\Omega^{(a)}(e) < 1$ and let $\omega_1, \ldots, \omega_n$ be a basis of $J$ over the ring of rational integers. Then $e^{-m} = \xi \sum \omega_i$, with rational integral $\xi$, can be written as a sum of a finite number of terms $\pm \omega_i$; hence $1 = e^{m} \cdot e^{-m}$ is a sum of terms $e^{m} \pm \omega_i$.

$$\Omega^{(a)}, W(1) < \max \{\Omega^{(a)}(e) \cdot W(\pm \omega_i)\} < 1$$

for sufficiently large $m$. Now the result follows by Lemma 1.1. $\Omega_p, \Omega_q = \Omega_p$ follows as for Table 1.

$\Omega_p, \Omega_q = U$. There exist elements $\xi$ and $\eta$ in $J$ such that

$$\xi + \gamma = 1, \xi = 0 \pmod{p}, \eta = 0 \pmod{q}.$$  \hspace{1cm} (4)

Hence

$$\Omega_p, \Omega_q(1) < \max \{\xi \cdot \eta \cdot \gamma \} < 1.$$  \hspace{1cm} (5)

If $p^r \mid a$, then, since $a = a \eta + ay$, $\Omega_p, W_0(a) \leq \max \{\xi, \cdot \eta, \gamma \} \rightarrow 0$ as $m \rightarrow \infty$.

On the other hand, if $p ^r \mid a$, then $x_i \gamma_i \equiv 0 \pmod{p^r}$ for at least one term of any decomposition (3) of $a$ in the ring $J$, and so

$$\Omega_p(x_i) W_0(y_i) = \Omega_p(x_i) \cdot \Omega_p(y_i).$$

Evidently equality holds for the decomposition $a = 1a$. In the limiting cases $a = 0$, $J$ the relation reduces to $\Omega_p \cdot W_0 = \Omega_p$, $\Omega_p \cdot U = U$, respectively.

**Table 6**

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<tr>
<th>$W_1 \circ W_2$</th>
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<td>$\Omega^{(a)}$</td>
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<td>$W_a + b$</td>
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<tr>
<td>$W_b$</td>
<td>$U$</td>
<td>$W_b$</td>
<td>$U$</td>
<td>$V^{(a)}_p$</td>
<td>$V^{(b)}_p$</td>
<td>$W_a$</td>
<td>$W_a + b$</td>
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**Table 6.** The proofs are as for Table 5 with the following exceptions:

$\Omega^{(a)} \circ \Omega^{(a)} = \Omega^{(a)}$ follows as in Table 2.
\[\Omega^{(1)} \circ \Omega^{(2)} = U.\] There is a unit \(e\) of \(f\) such that \(\Omega^{(3)}(e) < 1\), but \(\Omega^{(3)}(e) > 1\); then the result follows with the help of Lemma 1.1.

\(\Omega^{(1)} \circ W = U\), \(W = \Omega_p\) or \(W_{a}\). Take a unit \(e\) such that \(\Omega^{(1)}(e) < 1\) and write \(e^{-m} = \sum \xi_i \omega_i\), where \(\omega_i\) is a basis of \(f\) and the \(\xi_i\) are rational integers. Then, since \(W\) is non-archimedean,

\[\Omega^{(1)} \circ W(1) < \sum \xi_i \omega_i(m) W(\xi_i \omega_i) \leq \Omega^{(2)}(e)^m. (\Sigma_i W^{(0)}(\omega_i)) < 1\]

for sufficiently large \(m\).

For Table 7 we define \(\Sigma_{p,q}\) as \(\max(\Omega_p, \Omega_q)\) and put

\[W^a_p(a) = \begin{cases} c^{-r} p^a(a) & \text{if } a \equiv 0 \text{ (mod } a) \\ 1 & \text{otherwise} \end{cases}\]

where \(r\) is determined by \(y_p \mid a\). By \(e\) we understand again a unit of \(f\) such that \(\Omega^{(1)}(e) < 1, \Omega^{(2)}(e) > 1\).

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<tr>
<th>(W_1 \times W_2)</th>
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<th>(\Omega^{(1)})</th>
<th>(\Omega_p)</th>
<th>(\Omega_q)</th>
<th>(W_a)</th>
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</table>

\(\Omega^{(1)} \times W_a = W_a\). If \(a \equiv 0 \text{ (mod } a)\), then, since \(a = e^m e^{-ma}\),

\[\Omega^{(1)} \times W_a(a) < \max(\Omega^{(1)}(e)^m, W_a(e^{-ma})) \to 0,\]

as \(m \to \infty\). If \(a \not\equiv 0 \text{ (mod } a)\), then, in every decomposition (3) of \(a\) (in \(f\)), \(x_i y_i \not\equiv 0 \text{ (mod } a)\) for some \(i\), hence

\[\max(\Omega^{(1)}(x_i), W_a(y_i)) > 1;\]

and the lower bound 1 is attained for the decomposition \(a = e^m e^{-ma}\).

\(\Omega^{(1)} \times \Omega_a = \Omega_a\). For every decomposition (3) of \(a\) (in \(f\)), \(\Omega_a(a) < \max(\Omega_a(x_i), \Omega_a(y_i))\), hence

\[\Omega_a(a) < \Omega^{(1)}(\Omega_a(a)).\]

Conversely, since \(a = e^m e^{-ma}\),

\[\Omega^{(1)}(\Omega_a(a)) < \max(\Omega^{(1)}(e^m), \Omega_a(e^{-ma})) < \Omega_a(e^{-ma}) = \Omega_a(a),\]

when \(m\) is sufficiently large.

\(\Omega^{(1)} \times \Omega^{(2)} = U\). Write \(1 = e^m e^{-2m}\) and express \(e^{-2m}\) as a sum of terms \(\pm \omega_i\), \(\Omega^{(1)} \times \Omega^{(2)}(1) < \max(\Omega^{(1)}(e^m), \Omega^{(2)}(e^{-m})) < 1\) \((n > 1)\).

\(\Omega_p \times \Omega_{a} = \Sigma_{p,q}\). For any decomposition (3), \(\Omega_p(a) < \max(\Omega_p(x_i), \Omega_q(y_i))\), hence

\[\Sigma_{p,q} = \max(\Omega_p(a), \Omega_q(a)) < \max(\Omega_p(x_i), \Omega_q(y_i)),\]

and so \(\Sigma_{p,q} < \Omega_p \times \Omega_q\). For the converse, let \(m\) be any positive integer and choose \(\xi\) and \(\eta\) in \(f\) such that \(\xi + \eta = 1, \xi = 0 \text{ (mod } p^m), \eta = 0 \text{ (mod } q^m)\).

Then \(\Omega_p \times \Omega_q(a) < \max(\Omega_p(x_i), \Omega_q(y_i), \Omega_p(a))\)

\[< \max(\Omega_p(a), \Omega_q(a))\],

when \(m\) is sufficiently large.

\(\Omega_p \times W_a = W^a_p\). Clearly \(\Omega_p \times W_a(a) = 1, a \not\equiv 0 \text{ (mod } a)\), so suppose \(a \equiv 0 \text{ (mod } a)\). Let \(p^r \mid a\) and \(p^s \mid a\) and put \(a = p^{r+c}\), so that \(s > r\) and \(p \not\equiv c\). It suffices to consider decompositions (3) of \(a\) in which all \(y_i = 0 \text{ (mod } a)\). Then at least one \(x_i\) satisfies \(p^{s-r-1} \mid x_i\), whence

\[\Omega_p \times W_a(a) > c^{-r} \Omega_p(a)\]

To show that equality is attained we construct a special decomposition of \(a\) as follows. Let \(u_1, \ldots, u_n\) be a basis of \(p^r\) and \(v_1, \ldots, v_n\) a basis of \(p^{s-r}\). Then \(a\) can be written

\[a = \Sigma_{i=1}^n \Sigma_{j=1}^n \xi_i \omega_i v_i\]

where the coefficients \(\xi_i\) are rational integers. Now for any positive integer \(m\) determine \(\pi\) and \(\gamma\) in \(f\) such that \(\pi + \gamma = 1, \pi = 0 \text{ (mod } p^m), \gamma = 0 \text{ (mod } c)\).

Then

\[a = \pi a + \gamma a = \pi a + \Sigma \xi_i v_i, \gamma u_i.\]
By hypothesis \( W_a(a) \) and \( W_a(\gamma \alpha) \) vanish, \( \Omega \Sigma \beta \xi \gamma = c^{-\epsilon} \Omega \Omega \alpha \) and \( \Omega \alpha \rightarrow 0 \) as \( m \rightarrow \infty \); hence \( \Omega \beta \times W_a(a) < c^{-\epsilon} \Omega \Omega \alpha \). We note again the limiting cases: \( \Omega \beta \times W_0 = W_0 \), \( \Omega \beta \times U = \Omega \beta \). 
\( W_a \times W_b = W_{a+b} \). Clear (Cf. § 21).

For Table 8 we define \( W_{a,b} = W_{a+b} + \min(W_a, W_b) \); further we abbreviate \( \Omega \alpha \times \Omega \alpha \) by \( \Phi \alpha \). This function is specified in greater detail in the discussion following the Table.

<table>
<thead>
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<th>Table 8.</th>
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<tr>
<td>( W_1 \otimes W_2 )</td>
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<tr>
<td>( W_1 )</td>
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<tr>
<td>( \Omega \alpha )</td>
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<tr>
<td>( W_a )</td>
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| \( \Omega \alpha \) | \( \Omega \beta \) | \( \Omega \gamma \) | \( \Omega \delta \) | \( \Omega \epsilon \) | \( \Omega \zeta \) | \( \Omega \eta \) | \( \Omega \theta \) | \( \Omega \iota \) | \( \Omega \kappa \) | \( \Omega \lambda \) | \( \Omega \mu \) | \( \Omega \nu \) |
| \( W_a \) | \( W_b \) | \( W_c \) | \( W_d \) | \( W_e \) | \( W_f \) | \( W_g \) | \( W_h \) | \( W_i \) | \( W_j \) | \( W_k \) | \( W_l \) | \( W_m \) |

\( \Omega \beta \otimes \Omega \beta = \Omega \beta \). As for Table 4.
\( \Omega \beta \otimes W_2 = W_2 \). Suppose first that \( a \equiv 0 \) (mod \( a \)). Then in every decomposition (3) of \( a \) \( \Omega \beta(x_i) > \Omega \beta \) for at least one suffix \( i \), and \( W_2(y_i) = 1 \) for at least one suffix \( j \), hence \( \Omega \beta \otimes W_2(a) > \Omega \beta + 1 \), and equality may hold, as the decomposition \( a = a_1 \). 1 shows.

Now let \( a \mid a \). In every decomposition (3) of \( a \) either \( a \equiv 0 \) (mod \( a \)) for at least one \( i \), and so \( W_2(y_i) = 1 \), whence \( \Omega \beta \otimes W_2(a) > 1 > c_{a}^\beta \Omega \beta \).

or \( a \equiv 0 \) (mod \( a \)) for all \( i \), and then at least one \( x_i \) satisfies the inequality \( \Omega \beta(x_i) > c_{a}^\beta \Omega \beta \), so that in any case
\( \Omega \beta \otimes W_2(a) > W_2(a) \). (6)

To show that equality holds in (6), we construct again a special decomposition of \( a \). Let \( \psi \mid \alpha \) and denote by \( \beta \) the positive rational prime divisible by \( p \). Let \( a \neq 0 \) be an arbitrary element of \( a \), and \( m \) any positive integer. Then there exists a second element \( \beta \) of \( a \) such that
\( a = (a \beta^m, \beta) \). (7)

Since \( a \beta \), there exist \( \mu \) and \( v \) in \( J \) such that \( a = \mu \beta^m, v \). (8)

It follows from this decomposition of \( a \) that
\( \Omega \beta \otimes W_2(a) < \Omega \beta(\mu \beta^m) + W_2(a) + \Omega \beta(v) + W_2(\beta) \). (9)

By (7) \( \psi \mid \beta \) and by (8) \( \Omega \beta(\psi \beta) = \Omega \beta(\psi \beta) \) for all sufficiently large \( m \), hence \( \Omega \beta(\mu \beta^m) = c_{a}^\beta \Omega \beta(a) \). Inserting this in (9) and letting \( m \) tend to infinity we obtain the required result.
\( W_a \otimes W_b = W_{a+b} \). Clear.

| Table 8. The proofs are as for Table 7 with the following exceptions: \( \Omega \alpha \times \Omega \alpha = \Phi \alpha \). Taking this as the definition of \( \Phi \alpha \), we find as for Table 4 that \( \Phi \alpha > 2 \Omega \alpha \). Let \( \Omega \alpha = a < 1 \). Then there exists an integer \( m \) such that
\( a = \frac{\Omega \alpha}{\Omega \alpha} = \frac{1}{a} \). It follows from the decomposition \( a = a \cdot m, a \cdot m \) that
\( \Phi \alpha(a) < \Omega \alpha(\cdot a) + \Omega \alpha(a, m) < \frac{2}{\sqrt[1/2]{a}} \Omega \alpha(a), \frac{1}{a} \).

Thus the pseudo-valuation \( \Phi \alpha \) is equivalent to \( \Omega \alpha \).)

\( \alpha \) in the terminology of § 14 we can say that \( \Phi \alpha \) is strongly equivalent to \( \Omega \alpha \). We do not attempt a more precise determination of \( \Phi \alpha \), as this is immaterial to our purpose, but we remark that in the special case when there exists a unit \( e \) such that
\( \Phi \alpha(a) \alpha = \Omega \alpha(a) \alpha \).

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| 14. Our next object is to establish a connexion between the binary operations defined in Ch. II and the unary operations defined in Ch. I. Whereas the formal laws proved in Ch. II (commutativity, associativity) took the form of equalities, we cannot hope to establish equalities now, but only some form of equivalence, since there is a certain amount of arbitrariness in the definitions.

We shall say that two admissible functions \( \phi_1, \phi_2 \) are equivalent, \( \phi_1 \sim \phi_2 \), if, for any sequence \( a_n \) in \( R \), \( \phi_1(a_n) \to 0 \) if and only if \( \phi_2(a_n) \to 0 \). The functions will certainly be equivalent if there exists a constant \( k \) such that
\( \phi_1 < k \phi_2 \) and \( \phi_2 < k \phi_1 \). |
In this case we say that \( \varphi_1 \) and \( \varphi_2 \) are strongly equivalent and write \( \varphi_1 \approx \varphi_2 \). We use the signs \( \sim \) and \( \not\sim \) to indicate the negation of \( \sim \) and \( \approx \) respectively.

The concept of equivalence is of greater theoretical importance than that of strong equivalence, because it determines the largest class of pseudo-valuations which define the same topology on \( R \).\(^{10}\) However, we shall often use the notion of strong equivalence, as it is easier to handle. This is largely due to the fact (which is easily verified) that a linear operation is always \( \approx \)-invariant.

15. The relations \( \sim \) and \( \approx \) are evidently reflexive, symmetric and transitive; thus \( \sim \) defines a partition of the set of all admissible functions on \( R \) into equivalence-classes, the \( \sim \)-classes, and each such class is further subdivided into \( \approx \)-classes. We shall say that an operation \( \ast : \varphi \to \varphi^\ast \) on admissible functions is \( \sim \)-invariant, if

\[ \varphi_1 \sim \varphi_2 \implies \varphi_1^\ast \sim \varphi_2^\ast. \]

The operation \( \ast \) is called \( \approx \)-invariant, if

\[ \varphi_1 \approx \varphi_2 \implies \varphi_1^\ast \approx \varphi_2^\ast. \]

A binary operation is called \( \sim \)-invariant \((\approx \)-invariant\), if it is \( \sim \)-invariant \((\approx \)-invariant\) with respect to each argument. Thus a \( \sim \)-invariant operation is one which can be defined in a natural way on the \( \sim \)-classes, and similarly for \( \approx \)-invariance.

We note that of the two properties, \( \sim \)-invariance and \( \approx \)-invariance, neither implies the other. Thus the operation which associates with each pseudo-valuation \( W \) the pseudo-valuation \( V \) defined by

\[ V(a) = [W(a)]^{1/2}, \]

is \( \sim \)-invariant but not always \( \approx \)-invariant. On the other hand, the non-archimedean product on a field is \( \approx \)-invariant but not always \( \sim \)-invariant, as we shall see later (§19).

We have the obvious

**Lemma 15.1.** The operations \( + \) and \( \oplus \) are \( \sim \)-invariant. This is clear, since both operations are linear.

On the other hand, the operation \( \times \) defined in §4. is not \( \approx \)-invariant,

nor are the combined operations \( \times + \) or \( \times \oplus \). For, consider any proper non-archimedean pseudo-valuation \( W \), and define \( \varphi \) by

\[ \varphi(a) = \frac{1}{2W(1)} W(a). \]

Then \( \varphi \) is admissible and non-archimedean, and \( \varphi \approx W \), but \( \varphi^\times = U \) by Lemma 1.1, because \( \varphi^\times(1) = \frac{1}{2} \). Hence \( \varphi^\times \sim W \) and a fortiori \( \varphi^\times \not\approx W \). Since \( \varphi^\times \) and \( W \) are non-archimedean pseudo-valuations, it follows also that \( \times + \) and \( \times \oplus \) are not \( \approx \)-invariant. The argument shows further that \( \times \) is not \( \sim \)-invariant.

16. Let \( W_1 \) and \( W_2 \in \Omega_R \). We shall prove in this § that the greatest non-archimedean (subadditive) pseudo-valuation majorised by \( W_1 \) and \( W_2 \) is strongly equivalent to the non-archimedean (subadditive) product of \( W_1 \) and \( W_2 \). We simply follow the construction of §§4—6.

**Lemma 16.1.** Let \( W_1 \) and \( W_2 \) be any two submultiplicative admissible functions and put

\[ \varphi(a) = \min(W_1(a), W_2(a)). \]  

Then

\[ \varphi^\times(a) = \inf_{xy = a} W_1(x)W_2(y). \]

**Proof.** Write

\[ \Psi(a) = \inf_{xy = a} W_1(x)W_2(y); \]

we have to show that \( \varphi^\times \approx \Psi \). From the factorisations \( a = a.1 = 1.a \) we see that \( \Psi(a) \leq W_1(a)W_2(1) \) and \( \Psi(a) \leq W_2(1)W_2(a) \), hence \( \Psi(a) \leq \kappa \rho(a) \), where \( \kappa = \max(W(1), W(1)) \). Therefore, if we can prove that \( \Psi \) is submultiplicative it will follow that

\[ \Psi(a) \leq \kappa \rho^\times(a). \]

Now

\[ \Psi(ab) = \inf_{xy = ab} W_1(x)W_2(y). \]

Hence, for any factorisations \( a = x'y'^\ast, b = y'y'\ast \) of \( a \) and \( b \) we have, since \( x'y'x'y'\ast = ab \),

\[ \Psi(ab) \leq W_1(x'y')W_2(x'y') \leq W_1(x'y')W_1(y'y)W_2(x'y')W_2(y'y). \]

Taking the lower bound with respect to all such factorisations, we obtain

\[ \Psi(ab) \leq \inf_{x'y' = a} \{W_1(x')W_2(x')\} \inf_{y'y' = b} \{W_1(y')W_2(y')\} = \Psi(a)\Psi(b). \]
Hence $\Psi$ is submultiplicative, and it follows that $\Psi < kp^x$.

Conversely, given any factorisation $a = xy$,
\[ \varphi^x(a) = \varphi^x(xy) \leq \varphi^x(x)\varphi^y(y) \leq \varphi(x)\varphi(y) \leq W_1(x)W_2(y); \]
hence $\varphi^x(a) \leq \inf_{xy=a} W_1(x)W_2(y) = \Psi(a)$. Thus $\varphi^x < \Psi < kp^x$ and the lemma follows.

**Lemma 16.2.** If $W_1$, $W_2 \in \Omega_R$ and $\Psi(a) = \inf_{a=xy} W_1(x)W_2(y)$,
\[ \Psi^+ = W_1 \circ W_2, \quad \Psi^\oplus = W_1 \ominus W_2. \]

**Corollary 1.** $W_1 \circ W_2 < W_1 \circ W_2$.

**Corollary 2.** If $W_1 \circ W_2$ is non-archimedean, then $W_1 \circ W_2 = W_1 \circ W_2$.

**Proof.** By definition
\[ \Psi^+(a) = \inf_{a=xy} \max_i \Psi(\pm z_i) \]
\[ = \inf_{a=xy} \max_i \inf_{z_i=\pm z_i} W_1(x_i)W_2(y_i), \]
\[ \text{or} \]
\[ \Psi^+(a) = \inf_{a=xy} \max_i \inf_{z_i=\pm z_i} W_1(x_i)W_2(y_i), \]
(3)
since $W_1$ is a pseudo-valuation. To complete the proof we use the following lemma on real functions in $R$.

**Lemma 16.3.** If $z_1, \ldots, z_n$ are fixed elements of $R$ and $F$ is a real-valued function of two variables in $R$, then
\[ \max_{i=1,\ldots,n} \inf_{z_i=\pm z_i} F(x_i, y_i) = \inf_{z_i=\pm z_i} \max_{i=1,\ldots,n} F(x_i, y_i). \]
(4)

For when we consider a fixed $i$, then
\[ \inf_{z_i=\pm z_i} F(x_i, y_i) \leq \inf_{z_i=\pm z_i} \max_i F(x_i, y_i), \]
and hence
\[ \max_i \inf_{z_i=\pm z_i} F(x_i, y_i) < \inf_{z_i=\pm z_i} \max_i F(x_i, y_i). \]
To obtain the reverse inequality, denote the value of the left-hand side by $K$. Given $\varepsilon > 0$, we can choose $x_i, y_i$ ($i = 1, \ldots, n$) such that $F(x_i, y_i) < K + \varepsilon$ ($i = 1, \ldots, n$), and this shows that
\[ \inf_{z_i=\pm z_i} \max_i F(x_i, y_i) < K + \varepsilon. \]
Letting $\varepsilon$ tend to zero we obtain (4), and this proves the lemma.

We apply this lemma to the decomposition (3) with $F(x, y) = W_1(x)W_2(y)$ and get
\[ \Psi^+(a) = \inf_{a=xy} \max_i W_1(x_i)W_2(y_i) = W_1 \circ W_2(a). \]

Similarly we have, by definition,
\[ \Psi^\oplus(a) = \inf_{a=xy} \Sigma_i \Psi(\pm z_i) \]
\[ = \inf_{a=xy} \Sigma_i \inf_{z_i=\pm z_i} W_1(x_i)W_2(y_i), \]
whence
\[ \Psi^\oplus(a) = \inf_{a=xy} \Sigma_i W_1(x_i)W_2(y_i) = W_1 \circ W_2(a), \]
and this completes the proof of Lemma 16.2. The two corollaries now follow from the properties of $+$ and $\oplus$ (§ 5).

Since $\Psi \leq \varphi^x$, where $\varphi(a) = \min(W_1(a), W_2(a))$, it follows from Lemma 15.1 that $\varphi^{x+} \approx W_1 \circ W_2$ and $\varphi^{x=} \approx W_1 \circ W_2$. By Theorem 6.1 we can express this as

**Theorem 16.4.** If $W_1$, $W_2 \in \Omega_R$, then the greatest pseudo-valuation (non-archimedean pseudo-valuation) majorized by $\min(W_1, W_2)$ is strongly equivalent to the subadditive product $W_1 \circ W_2$ (resp. the non-archimedean product $W_1 \cdot W_2$).

If $W_1(1), W_2(1) < 1$, then the strong equivalence in Theorem 16.4 can be replaced by equality and in this form the Theorem provides an alternative proof of the entries in the Tables 1, 2, 5 and 6 of Ch. III.

17. There is no analogous interpretation of the compound of two pseudo-valuations, but we can now define a function
\[ \Psi(a) = \inf_{a=xy} \max(W_1(x), W_2(y)) \]
(5)
and show that
\[ \Psi^+ = W_1 \times W_2, \]
\[ \Psi^\oplus \approx W_1 \otimes W_2. \]
(6)
(7)

By defining $\Psi$ as $\inf(W_1(x), W_2(y))$ instead of using (5) we can strengthen (7) to an equality but only at the cost of weakening (6) to an equivalence. We shall not go into the proof which is very similar to the case of the product given in the previous §, but only mention the results which here correspond to the corollaries of Lemma 16.2:

**Theorem 17.1.** $W_1 \times W_2 < W_1 \otimes W_2$.

**Theorem 17.2.** If $W_1 \otimes W_2$ is non-archimedean, then $W_1 \otimes W_2 \approx W_1 \times W_2$.

18. We now come to the question of the invariance of the different operations. For the sake of a later application we prove the invariance properties not with respect to equivalence, but with
respect to the quasi-ordering which defines the equivalence. For this purpose we recall the definition of P.I. 8:

If \( W_1, W_2 \in \Omega_p \), then \( W_1 \) is said to be \textit{contained} in \( W_2 \): \( W_1 \subseteq W_2 \), if to every \( \varepsilon_1 > 0 \) there corresponds a number \( \varepsilon_2 \) such that

\[
W_1(a) < \varepsilon_1 \text{ holds whenever } W_2(a) < \varepsilon_2. \tag{8}
\]

More briefly, the condition (8) states that \( W_1 \) is small whenever \( W_2 \) is small. Similarly we say that \( W_1 \) is strongly contained in \( W_2 \): \( W_1 \subseteq \sim W_2 \), if \( W_1 < kW_2 \) for some \( k \). It is clear that \( W_1 \subseteq \sim W_2 \) implies \( W_1 \subset W_2 \). Further \( W_1 \sim W_2 \) if and only if \( W_1 \subset W_2 \subset W_1 \); \( W_1 \approx W_2 \) if and only if \( W_1 \subseteq W_2 \subseteq W_1 \).

\textbf{Lemma 18.1} \ If \( W_1 \subseteq W_i \) (\( i = 1, 2 \)), then

\[ W_1 \ominus W_2 \subseteq W_1 \ominus W_2, \]

where \( \ominus \) is \( \cdot \), \( \odot \), \( \times \) or \( \otimes \).

\textit{Proof.} Let \( W_1 < kW_1', \ W_2 < kW_2' \), then it is clear from the definitions that

\[ W_1 \ominus W_2 \subseteq lW_1' \ominus W_2', \]

where \( l = \max \{k^2, 1\} \).

By applying this Lemma first as it stands, and then with \( W_1 \) and \( W_i \) interchanged (\( i = 1, 2 \)), we obtain

\textbf{Theorem 18.2.} The operations \( \cdot \), \( \odot \), \( \times \) and \( \otimes \) are \( \approx \)-invariant.

19. The results on \( \sim \)-invariance are less complete. For the non-archimedean operations they are given by the following theorems.

\textbf{Theorem 19.1.} The non-archimedean compound is \( \sim \)-invariant.

\textbf{Theorem 19.2.} The non-archimedean product is \( \sim \)-invariant when applied to bounded pseudo-valuations.

We prove these Theorems by a Lemma analogous to Lemma 18.1:

\textbf{Lemma 19.3.} If \( W_1 \subset W_i \) (\( i = 1, 2 \)), then \( W_1 \times W_2 \subset W_1' \times W_2' \); and further if \( W_1 \) and \( W_i \) are bounded, then \( W_1, W_2 \subset W_1' \times W_2' \).

\textit{Proof.} Consider first \( \times \). We shall prove: If \( W_1 \subset W_i' \), then \( W_1 \times W_2 \subset W_1' \times W_2 \). From this the first part of the Lemma will follow by the commutativity of \( \times \).

Assume that

\[ W_1' \times W_2(a) < a \quad (a > 0); \tag{9} \]

then there is a decomposition \( a = x_0y_1 + \ldots + x_ny_n \) of \( a \) such that \( \max \{W_1'(x_i), W_2'(y_i)\} < 2a \), or

\[ W_1'(x_i) < 2a, \ W_2'(y_i) < 2a \quad (i = 1, \ldots, n). \tag{10} \]

Let \( \varepsilon > 0 \) be fixed. Then there is a \( \delta > 0 \) such that

\[ W_1'(a) < \delta \text{ implies } W_1(a) < \varepsilon \text{ where } z \in R. \tag{11} \]

Now choose \( a = \frac{1}{2} \min(\delta, \varepsilon) \) in (9); then, by (10),

\[ W_2(y_i) < 2a < \varepsilon \text{ and } W_1'(x_i) < 2a < \delta, \]

so that \( W_1(x_i) < \varepsilon \) and therefore

\[ W_1 \times W_2(a) < \max\{W_1(x_i), W_2(y_i)\} < \varepsilon. \]

Hence \( W_1 \times W_2 \) is small whenever \( W_1' \times W_2 \) is small and this proves the first part of the Lemma.

The proof for the non-archimedean product is similar. We suppose now that \( W_1 \) and \( W_2 \) are bounded, say that

\[ W_1(z) < \omega, \ W_2(z) < \omega \quad \text{for all } z \in R. \]

If

\[ W_1' \cdot W_2(a) < a \quad (a > 0), \tag{12} \]

then there is a decomposition \( a = x_0y_1 + \ldots + x_ny_n \) of \( a \) such that \( \max\{W_1'(x_i), W_2'(y_i)\} < 2a \). Therefore for each \( i = 1, \ldots, n \),

\[ W_1'(x_i) < \sqrt{2a} \text{ or } W_2'(y_i) < \sqrt{2a}. \tag{13} \]

Let \( \delta = \delta(\varepsilon) \) be as in (11) and choose \( a = \frac{1}{2} \min(\delta(\varepsilon), 2a) \) in (12). Then, by (13), either \( W_1'(x_i) < \sqrt{2a} < \delta(\varepsilon) \) or hence \( W_1(x_i) < \delta \varepsilon, \) or \( W_2(y_i) < \sqrt{2a} < \delta \varepsilon. \) In either case

\[ W_1(x_i)W_2(y_i) < \delta \varepsilon \omega = \varepsilon, \]

whence

\[ W_1 \cdot W_2(a) < \max\{W_1(x_i)W_2(y_i)\} < \varepsilon. \]

This completes the proof of the Lemma.

We note that in the second part of the Lemma it is enough to assume that \( W_1 \) and \( W_2 \) are bounded for the conclusion to hold.

Theorems 19.1 and 19.2 are an immediate consequence of the Lemma. The following example shows that the boundedness condition in Theorem 19.2 cannot be omitted.

Let \( \Omega_p \) be the \( p \)-adic valuation on the rational field. Then \( \Omega_p^{1/2} \) is again a valuation and is equivalent to \( \Omega_p \). By the remark after Theorem 16.4, or by Table 1,

\[ \Omega_p^{1/2}, \Omega_p = \Omega_p. \]

On the other hand

\[ \Omega_p^{1/2}, \Omega_p = U. \]
For $\Omega_p(\rho) < \Omega_p^{1/2}(\rho)$, and so
\[
\Omega_p^{1/2}\cdot\Omega_p(1) < \Omega_p\left(\frac{1}{\rho}\right)^{1/2}\cdot\Omega_p(\rho)
\]
\[
= \Omega_p(\rho)^{1/2}\cdot\Omega_p(\rho) < 1.
\]
Thus $\Omega_p^{1/2}\cdot\Omega_p \sim \Omega_p\cdot\Omega_p$, although $\Omega_p^{1/2} \sim \Omega_p$. Similarly one can show that $\Omega_{p^2}\cdot\Omega_p = U$, and therefore $\Omega_{p^2}\cdot\Omega_p \sim \Omega_p\cdot\Omega_p$. It will be proved later (§ 22) that if only bounded and non-archimedean pseudo-valuations are considered, then the subadditive product is equivalent to the non-archimedean product and is therefore $\sim$-invariant. The example just given shows that the boundedness condition is again essential. The question of the $\sim$-invariance of the subadditive compound remains open.

V.

20. With every ideal $a$ of the ring $R$ a pseudo-valuation $W_a$ can be associated by defining

\[
W_a(a) = \begin{cases} 
0 & \text{if } a = 0 \pmod{a}, \\
1 & \text{otherwise} \quad (1)
\end{cases}
\]

This is consistent with the notation $W_0$ for the trivial pseudo-valuation, which can be regarded as the pseudo-valuation associated with the zero-ideal. Similarly the improper pseudo-valuation $U$ is associated with the whole ring.

In P.I. 3 it was shown that the set of elements for which a given pseudo-valuation vanishes is an ideal, and so we have the converse that every pseudo-valuation $W$ which takes only the values 0 and 1 must be the pseudo-valuation associated with an ideal $a$; here $a$ is uniquely determined as the set where $W$ vanishes.

Let us call a pseudo-valuation of the form (1) special. It is clear that two special pseudo-valuations which are equivalent must in fact be equal, so that the relation $\sim$ defines a partial ordering on the set of special pseudo-valuations. This corresponds to the partial ordering by inclusion of the ideals of $R$ in the sense that

\[
W_a \subset W_b \quad \text{if and only if } a \supseteq b. \quad (2)
\]

Therefore every relation between ideals in $R$ corresponds to a relation between special pseudo-valuations and conversely. We investigate some of these relations and in particular give another interpretation of the operations defined in Ch. II.

21. Let $a, b$ be any ideals of $R$ and $W_a, W_b$ the pseudo-valuations associated with them. Consider the subadditive product $V = W_a \cdot W_b$. From the definition it is clear that $V$ takes only the values 0 or 1 and is therefore special; and $V(c) = 0$ if and only if there is a decomposition $c = y_1 + \ldots + y_n$ for which $W_a(y_i)W_b(y_i) = 0$ ($i = 1, \ldots, n$), that is, if and only if $c \not\in a + b$. Hence

\[
W_a \cdot W_b = W_{a+b}.
\]

In the same way (or by Lemma 16.2 Cor. 2) it follows that

\[
W_a \cdot W_b = W_{a+b},
\]

so that the product of special pseudo-valuations corresponds to the sum of ideals.

Consider next $V' = W_a \times W_b$. Again $V'$ takes only the values 0 and 1. Further $V'(c) = 0$ if and only if there is a decomposition $c = y_1 + \ldots + y_n$ of $c$ such that $\max(W_a(y_i), W_b(y_i)) = 0$, which is the case if and only if $x_i \in a, y_i \in b$, i.e. if $c \in ab$. Hence $V' = W_{ab}$. If $c \in ab$, it is clear that $W_a \cdot W_b(c)$ also vanishes; combining this fact with the relation $W_a \times W_b < W_a \otimes W_b$ (Theorem 17.1) we see that the subadditive product is equivalent to $W_{ab}$. Now $W_a \otimes W_b$ is bounded by the constant 2, as we see by using the decomposition $c = c.1$; hence $W_a \otimes W_b$ is strongly equivalent to $W_{ab}$. Summing up, we have

**Theorem 21.1.** If $W_a, W_b$ are the pseudo-valuations associated with two ideals $a, b$ of $R$, then

\[
W_a \otimes W_b = W_a \cdot W_b = W_{a+b},
\]

\[
W_a \cdot W_b \approx W_a \times W_b = W_{ab}.
\]

The strong equivalence in the last line cannot be improved to equality since e.g. $W_a \otimes W_a(1) = 2$. In fact, as we saw in Ch. III, $W_a \otimes W_b = W_{ab} + \min(W_a, W_b)$.

For the sake of comparison we recall the result proved in P.I. 11 that $\max(W_a, W_b) = W_{a+b}$. We restate this in a more general form as

\[\text{Footnote:} \]
Theorem 21.2. If \( a_\lambda (\lambda \in A) \) is any non-empty family of ideals and their intersection is \( a \), then
\[
\sup \{ W_\lambda a \mid \lambda \in A \} = W_a.
\]
This Theorem, together with (2), implies that the set of ideals of \( R \) and the set of special pseudo-valuations form isomorphic complete lattices.

22. The interpretation of the product of two special pseudo-valuations given in § 21 can be extended to the case of bounded non-archimedean pseudo-valuations.

Let \( W_1, W_2 \) be two such pseudo-valuations and consider an element \( a \) of \( R \) for which \( W_1, W_2(a) \) is small. If \( W_1, W_2(a) < a \), where \( a > 0 \), then there is a decomposition \( a = x_1 y_1 + \ldots + x_n y_n \) of \( a \) such that
\[
W_1(x_i) W_2(y_i) < 2a \quad (i = 1, \ldots , n)
\]
Let \( \omega \) be an upper bound for \( W_1 \) and \( W_2 \), then for each \( i = 1, \ldots , n \) either \( W_1(x_i) < \sqrt{2a} \) or \( W_2(y_i) < \sqrt{2a} \), and so either
\[
W_1(x_i y_i) < W_1(x_i) W_2(y_i) < \omega \sqrt{2a} \quad (3)
\]
or
\[
W_2(x_i y_i) < W_2(x_i) W_2(y_i) < \omega \sqrt{2a}. \quad (4)
\]
Denote by \( b \) the sum of the terms \( x_i y_i \) for which (3) holds, and by \( c \) the sum of the remaining terms. Then \( a = b + c \), and by (3) and (4)
\[
W_1(b) < \omega \sqrt{2a}, \quad W_2(c) < \omega \sqrt{2a},
\]
since \( W_1 \) and \( W_2 \) are non-archimedean. Thus if \( \varepsilon > 0 \) is fixed, and \( \delta < \frac{1}{2}(\varepsilon/\omega)^2 \), then for any \( a \in R \) such that \( W_1, W_2(a) < \delta \), we can find \( b \) and \( c \in R \) such that \( b + c = a \) and \( W_1(b) < \varepsilon, W_2(c) < \varepsilon \).

More concisely, we can say that if \( W_1, W_2(a) \) is small, then \( a = b + c \), where \( W_1(b) \) and \( W_2(c) \) are small.

Conversely, if \( W_1(b) < a \), \( W_2(c) < a \), then by Theorem 16.4
\[
W_1 W_2(b + c) \leq \max(W_1 W_2(b), W_1 W_2(c)) < k \max(W_1(b), W_2(c)) < k a,
\]
where \( k \) depends on \( W_1 \) and \( W_2 \) only. By taking \( \delta < \varepsilon/k \), we see that if \( W_1(b), W_2(c) < \delta \), then \( W_1 W_2(b + c) < \varepsilon \), and so we have obtained the following result:

---

non-archimedean pseudo-valuations. As examples we consider the ideal relations

1) \( a \cap b \supseteq ab \),
2) \( a(b + c) = ab + ac \),
3) \( ab \supseteq (a \cap b)(a + b) \).

The corresponding relations for pseudo-valuations are given by

**Theorem 23.1.** If \( W_1, W_2, W_3 \) are bounded non-archimedean pseudo-valuations, then

1) \( W_1 + W_2 \subseteq W_1 \times W_2 \),
2) \( W_1 \times (W_2, W_3) \sim (W_1 \times W_2, W_1 \times W_3) \),
3) \( W_1 \times W_2 \subseteq (W_1 + W_2) \times (W_1, W_2) \).

**Proof.** 1) Let \( \omega \) be an upper bound for \( W_1 \) and \( W_2 \). If \( W_1 \times W_2(a) < a \) \((a > 0)\), then there is a decomposition \( a = x_1 y_1 + \ldots + x_n y_n \) such that \( W_1(x_i), W_2(y_i) < 2 \omega \). Hence

\[ W_1(a) < \max\{W_1(x_i)W_2(y_i)\} < 2 \omega \omega. \]

Similarly \( W_2(a) < 2 \omega \omega \), and 1) follows.

2) Since \( W_2, W_3 \subseteq W_2 \), it follows by Lemma 18.1 that

\( W_1 \times (W_2, W_3) \subseteq W_1 \times W_2 \) and similarly \( W_1 \times (W_2, W_3) \subseteq W_1 \times W_3 \); in other words

\( W_1 \times (W_2, W_3) \subseteq k(W_1 \times W_2), k(W_1 \times W_3) \),

where \( k \) is some constant, which may without loss of generality be taken to be greater than 1. Then \( k(W_1 \times W_2) \) is again a pseudo-valuation, hence by Theorem 16.4

\[ W_1 \times (W_2, W_3) \subseteq [k(W_1 \times W_2), [k(W_1 \times W_3)]] \subseteq (W_1 \times W_2), (W_1 \times W_3); \]

therefore the left-hand side of 2) is small if the right-hand side is small.

Conversely, if \( W_1 \times (W_2, W_3) \) is small, then a can be written as \( a = x_1 y_1 + \ldots + x_n y_n \), where \( W_1(x_i) \) and \( W_2(y_i) \) are small. Hence \( y_i \) has the form \( y_i = y_i' + y_i'' \), where \( W_2(y_i') \) and \( W_3(y_i') \) are small. Then \( W_1 \times W_2(\Sigma x_i y_i') \) and \( W_1 \times W_3(\Sigma x_i y_i'') \) are small, and since \( a = \Sigma x_i y_i' + \Sigma x_i y_i'' \), it follows that \( W_1 \times W_3(W_1 \times W_3)(a) \) is small, and this proves 2).

3) By Lemma 19.3, since \( W_1, W_2 \subseteq W_1 + W_2 \),

\[ W_1 \times W_2 \subseteq (W_1 + W_2) \times W_1, \]
\[ W_1 \times W_2 \subseteq (W_1 + W_2) \times W_2; \]

hence by Theorem 16.4

\[ W_1 \times W_2 \subseteq [(W_1 + W_2) \times W_1][W_1 + W_2] \times W_2 \]
\[ \subseteq (W_1 + W_2) \times (W_1, W_2) \]

by 2). This completes the proof.

24. It would be of interest to decide whether any relation holding between ideals in a commutative ring with a unit-element always has its analogue for non-archimedean bounded pseudo-valuations. The converse is obviously true: If a relation holds generally for non-archimedean bounded pseudo-valuations, then the corresponding relation for ideals must also hold, since all we need is to consider the special pseudo-valuations in order to effect the changeover to ideals. As an illustration of this principle we prove that the product or the compound of pseudo-valuations is not distributive with respect to addition.

**Theorem 24.1.** The equivalences

\[ W_1 \cdot (W_2 + W_3) \sim W_1 \cdot W_2 + W_1 \cdot W_3, \]
\[ W_1 \times (W_2 + W_3) \sim W_1 \times W_2 + W_1 \times W_3, \]

where the \( W_i \) are non-archimedean bounded pseudo-valuations, do not hold generally.

**Proof.** Suppose that (5) and (6) are true generally in any commutative ring with a unit-element. On taking the \( W_i \) to be special and rewriting the resulting equations in terms of ideals we obtain the equations

\[ a + b \cap c = (a + b) \cap (a + c), \]
\[ a \cdot (b \cap c) = (a b) \cap (a c), \]

which must hold identically in the ideals \( a, b \) and \( c \). It is therefore enough to give examples of rings in which these equations are false.

i) Let \( R \) be an algebra over any field with basis \( 1, u, v \), and multiplication-table

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\( R \) is associative and commutative and has a unit-element. Denote by \( a, b \) and \( c \) the ideals generated by \( u + v, u \) and \( v \) respectively. Then

\[ a + (b \cap c) = (u + v) \neq (u, v) = (a + b) \cap (a + c). \]
ii) Let $R$ be the ring of polynomials in two indeterminates $x, y$ over a field, and put $a = (x, y), \ b = (x^2), \ c = (y^2)$.

Then $a(b \cap c) = (x^3y^2, x^2y^2) \neq (x^2y^2) = (ab) \cap (ac)$.

Therefore the equivalences (5) and (6) do not hold universally. This completes the proof.

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