ON COMPOUND CONVEX BODIES (II)

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The inequality

\[ 0 < c_1 \leq V(K)^{-p} V(K) \leq c_2 \]

for the volume of the \( p \)th compound \( K = [K]^{(p)} \) of a single convex body, which was proved in the first part of this paper, cannot be fully extended to the compounds \( K = [K^{(1)}, K^{(2)}, \ldots, K^{(p)}] \) of several distinct convex bodies. The problem of estimates for \( V(K) \) will be investigated in the present note, and a partial result will be proved.

1. We use the same notation as in the first part. As before, let

\[ K^{(1)}, K^{(2)}, \ldots, K^{(p)} \]

be any \( p \) closed, bounded, symmetric, convex bodies in \( R_n \), and let

\[ K = [K^{(1)}, K^{(2)}, \ldots, K^{(p)}] \]

be their compound in \( R_N \). We wish either to find upper and lower bounds for the volume \( V(K) \) in terms of some symmetric function of

\[ V(K^{(1)}), V(K^{(2)}), \ldots, V(K^{(p)}) \]

or to show that such bounds do not exist. Since, for positive \( t_1, t_2, \ldots, t_p \),

\[ [t_1 K^{(1)}, t_2 K^{(2)}, \ldots, t_p K^{(p)}] = t_1 t_2 \ldots t_p K, \]

and since further

\[ V(t_1 t_2 \ldots t_p K) = (t_1 t_2 \ldots t_p)^N V(K), \]

\[ V(t_1 K^{(1)}) = t_1^n V(K^{(1)}), \quad \ldots, \quad V(t_p K^{(p)}) = t_p^n V(K^{(p)}), \]

it is clear, for reasons of homogeneity, that we must compare \( V(K) \) with the expression

\[ \left( \prod_{\pi=1}^{P} V(K^{(\pi)}) \right)^{P/p}. \]

The question is therefore whether

\[ S(K) = V(K) \left( \prod_{\pi=1}^{P} V(K^{(\pi)}) \right)^{-P/p} \]

possesses positive upper and lower bounds depending only on \( n \) and \( p \).

2. For the upper bound, the problem is solved by the following theorem.

**Theorem 1.** Let \( n \geq 3 \) and \( 2 \leq p \leq n-1 \), and let \( c > 0 \) be arbitrary. Then there exist \( p \) closed, bounded, symmetric, convex bodies \( K^{(1)}, K^{(2)}, \ldots, K^{(p)} \) such that their compound \( K = [K^{(1)}, K^{(2)}, \ldots, K^{(p)}] \) satisfies the inequality

\[ S(K) > c. \]

Thus \( S(K) \) admits of no upper bound depending only on \( n \) and \( p \).
Proof. We choose for \( K^{(1)} = K^{(2)} = ... = K^{(p-1)} \) the generalized octahedron
\[
|x_1| + |x_2| + ... + |x_n| \leq 1,
\]
and for \( K^{(p)} \) the generalized octahedron
\[
\frac{1}{a} \left( |x_1| + |x_2| + ... + |x_{n-1}| \right) + a^{n-1}|x_n| \leq 1,
\]
where \( a \) is a parameter satisfying \( 0 < a \leq 1 \). Then
\[
V(K^{(1)}) = V(K^{(2)}) = ... = V(K^{(p)}) = \frac{2^n}{n!}.
\]
The first \( p-1 \) octahedra have the vertices
\[
\pm U_1, \pm U_2, ..., \pm U_n,
\]
where
\[
U_1 = (1, 0, ..., 0), \quad U_2 = (0, 1, ..., 0), \quad ..., \quad U_n = (0, 0, ..., 1)
\]
denote the \( n \) unit points on the coordinate axes in \( R_n \). Similarly the vertices of the last octahedron lie at
\[
\pm aU_1, \pm aU_2, ..., \pm aU_{n-1}, \pm a^{-(n-1)}U_n.
\]
The compound body \( K = [K^{(1)}, K^{(2)}, ..., K^{(p)}] \) contains, in particular, the convex hull \( H \) of the \( 2N \) compound points
\[
\pm a_{v_p} [U_{v_1}, U_{v_2}, ..., U_{v_p}]. \tag{1}
\]
Here \( v_1, v_2, ..., v_p \) run over all \( \mathcal{N} \) distinct sets of \( p \) indices satisfying
\[
1 \leq v_1 < v_2 < ... < v_p \leq n,
\]
and we have, for shortness, put
\[
a_{v_p} = a \text{ if } v_p = 1, 2, ..., n-1, \quad \text{but} \quad a_{v_p} = a^{-(n-1)} \text{ if } v_p = n.
\]
The \( \mathcal{N} \) compounds \([U_{v_1}, U_{v_2}, ..., U_{v_p}]\) coincide with the unit points on the \( \mathcal{N} \) coordinate axes in \( R_N \); evidently exactly \( P \) of them belong to \( v_p = n \). Hence all points \( (1) \) lie on the coordinate axes; just \( 2P \) of them have one coordinate equal to \( \pm a^{-(n-1)} \) and the other coordinates equal to zero; and each of the remaining \( 2(N-P) \) points has just one coordinate equal to \( \pm a \) and the other coordinates equal to zero. Thus, if the numbering of the coordinates \( \xi_1, \xi_2, ..., \xi_N \) of the general point \( \Xi \) in \( R_N \) is chosen suitably, then the convex hull \( H \) of the points \( (1) \) becomes the generalized octahedron
\[
\frac{1}{a} \sum_{k=1}^{N-P} |\xi_k| + a^{n-1} \sum_{k=N-P+1}^{N} |\xi_k| \leq 1
\]
of volume
\[
V(H) = \frac{2^N}{N!} a^{(N-P)-(n-1)P}.
\]
Since \( H \) is a subset of \( K \), this implies that
\[
V(K) \geq \frac{2^N}{N!} a^{(N-P)-(n-1)P},
\]
and we therefore obtain the inequality
\[ S(K) = V(K) \left\{ \prod_{\pi=1}^{p} V(K^{(\pi)}) \right\}^{1/P} \geq \frac{2^{N}}{N!} \left( \frac{2^{n}}{n!} \right)^{-P} a^{(N-P)-(n-1)P}. \]
Here the expression on the right-hand side can be made arbitrarily large by choosing \( a \) sufficiently small because
\[(N-P)-(n-1)P = N-nP = \binom{n}{p} n \binom{n-1}{p-1} = -\frac{n(p-1)}{p} \binom{n-1}{p-1} < 0.\]
This proves the assertion.

3. It is much more difficult to decide whether \( S(K) \) possesses any positive lower bound that depends only on \( n \) and \( p \). In this note the problem will be settled in the special case when \( n \geq 3 \), \( 2 \leq p \leq n-1 \), and when \( K^{(1)}, K^{(2)}, \ldots, K^{(p)} \) are made up by repetition of just two distinct convex bodies.

To fix the notation, let \( p = r+s \), \( r \geq 1 \), \( s \geq 1 \); assume that the first \( r \) of the bodies \( K^{(1)}, K^{(2)}, \ldots, K^{(p)} \) are identical with \( K_1 \), and that the last \( s \) bodies are identical with \( K_2 \). We then write, for shortness,
\[ K = [K_1^r K_2^s], \]
and the number \( S(K) \) takes the form
\[ S(K) = V(K) \{V(K_1)^r V(K_2)^s\}^{-P/p}. \]
We have to show that \( S(K) \) is not smaller than a certain positive number which depends only on \( n \) and \( p \).

4. Let us begin with the simpler case when \( K_1 = E_1 \) and \( K_2 = E_2 \) are ellipsoids in \( R^N \) with centres at the origin. By the theory of reduction to principal axes for such ellipsoids, there exists a non-singular affine transformation \( X \to X' = \Omega X \) of \( R^n \) such that
\[ E_1 = \Omega G_n, \quad E_2 = \Omega E. \]
Here \( G_n \) denotes the unit sphere
\[ x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1, \]
and \( E \) is an ellipsoid of the form
\[ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \ldots + \frac{x_n^2}{a_n^2} \leq 1, \]
the semi-axes of which may be assumed to satisfy the inequalities
\[ 0 < a_1 \leq a_2 \leq \ldots \leq a_n. \tag{2} \]
Evidently
\[ K = [E_1^r E_2^s] = \Omega^{(p)} [G_n^r E^s], \]
where \( \Omega^{(p)} \) is the \( p \)th compound of \( \Omega \). Denote by \( \omega \) the determinant of \( \Omega \); then
\[ \omega^{(p)} = \omega^P. \]
is the determinant of $\Omega^{(p)}$. Let further
\[ K_0 = [G_n^p E^s]; \]
hence
\[ K = \Omega^{(p)} K_0. \]

The volumes of $G_n$ and $E$ are given by
\[ V(G_n) = \kappa_n, \quad V(E) = \kappa_n a_1 a_2 \ldots a_n, \]
where $\kappa_n$ is the constant
\[ \kappa_n = \frac{\pi^{\frac{1}{2n}}}{\Gamma(\frac{1}{2}n + 1)}. \]
Therefore also
\[ V(E_1) = \kappa_n |\omega|, \quad V(E_2) = \kappa_n a_1 a_2 \ldots a_n |\omega|. \]
On the other hand,
\[ V(K) = |\omega^{(p)}| V(K_0) = |\omega|^{P} V(K_0). \]
Hence
\[ S(K) = |\omega|^{P} V(K_0) \left((\kappa_n |\omega|)^r (\kappa_n a_1 a_2 \ldots a_n |\omega|)^s\right)^{-P/p}, \]
and this may be simplified to
\[ S(K) = \frac{V(K_0)}{\kappa_n^{P} (a_1 a_2 \ldots a_n)^{sP/p}}. \]

Denote again by $v_1, v_2, \ldots, v_p$ all $N$ sets of $p$ indices satisfying
\[ 1 \leq v_1 < v_2 < \ldots < v_p \leq n, \]
and, for each such set, put
\[ A(v) = a_{v_1} a_{v_2} \ldots a_{v_r}, \quad B(v) = a_{v_{r+1}} a_{v_{r+2}} \ldots a_{v_{r+s}}. \]
The product
\[ \prod_v (A(v)B(v)) = \prod_v (a_{v_1} a_{v_2} \ldots a_{v_p}) \]
extended over all $N$ sets is easily seen to be equal to
\[ (a_1 a_2 \ldots a_n)^P. \]
On the other hand, the hypothesis (2) gives the inequalities
\[ B(v) \geq (A(v)B(v))^{s/p}, \]
and it follows therefore that
\[ \prod_v B(v) \geq (a_1 a_2 \ldots a_n)^{sP/p}. \]

We can now derive a lower bound for $V(K_0)$; it would be much harder to determine the exact value of this number.

The unit sphere $G_n$ contains the $2n$ positive and negative unit points
\[ \pm U_1, \pm U_2, \ldots, \pm U_n, \]
and the ellipsoid $E$ contains the proportional points
\[ \pm a_1 U_1, \pm a_2 U_2, \ldots, \pm a_n U_n. \]
Hence $K_0 = [G_n^p E^s]$ contains all the compound points
\[ \pm B(v) [U_{v_1}, U_{v_2}, \ldots, U_{v_p}]. \]
Apart from the numerical factors \( \pm B(v) \), these points are just all the \( N \) distinct unit points on the coordinate axes in \( R_N \). Hence the convex hull \( H \) of the points (5) is a generalized octahedron of volume
\[
V(H) = \frac{2^N}{N!} \prod_v B(v).
\]

Since \( K_0 \supseteq H \), it follows then by (4) that
\[
V(K_0) \geq \frac{2^N}{N!} \prod_v B(v) \geq \frac{2^N}{N!} (a_1a_2 \ldots a_n)^{s/p^p}.
\]

We finally substitute this lower bound for \( V(K_0) \) in (3) and obtain the estimate
\[
S(K) \geq \frac{2^N}{\kappa^{pN} N!}.
\]
As asserted, the constant on the right-hand side depends only on \( n \) and \( p \).

5. The already asserted result can now be proved.

**Theorem 2.** Let \( n \geq 3 \), \( 2 \leq p \leq n-1 \), \( p = r+s \), \( r \geq 1 \), \( s \geq 1 \). Let further \( K_1 \) and \( K_2 \) be two closed, bounded, symmetric, convex bodies in \( R_n \), and let \( K = [K_1^r K_2^s] \) be a mixed compound of these bodies in \( R_N \). A positive constant \( c \) depending only on \( n \) and \( p \) exists such that
\[
V(K) \geq c \{(V(K_1)^r V(K_2)^s)^{p/p^p}. \]

**Proof.** By the theorem of John (1) there exist two ellipsoids \( E_1 \) and \( E_2 \) in \( R_n \) with their centres at the origin such that
\[
n^{-1/2} E_1 \subseteq K_1 \subseteq E_1, \quad n^{-1/2} E_2 \subseteq K_2 \subseteq E_2.
\]
Hence, if \( K_1 \) is the compound body
\[
K_1 = [E_1^r E_2^s],
\]
then
\[
n^{-1/p} K_1 \subseteq K \subseteq K_1,
\]
and so also
\[
n^{-1/p} V(K_1) \leq V(K) \leq V(K_1).
\]

It has already been proved that
\[
V(K_1) \geq \frac{2^N}{\kappa^{pN} N!} \{V(E_1)^r V(E_2)^s \}^{p/p^p}.
\]

Hence it follows from the left-hand inequality in (6) that
\[
V(K) \geq \frac{2^N}{\kappa^{pN} N!} n^{1/p} \{V(E_1)^r V(E_2)^s \}^{p/p^p} = \frac{2^N}{\kappa^{pN} N!} n^{1/p} \{V(K_1)^r V(K_2)^s \}^{p/p^p},
\]
as was to be proved.

**Reference**

1. F. John, Courant anniversary volume (1948), 187–204.

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