ON THE TAYLOR COEFFICIENTS OF RATIONAL FUNCTIONS

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Let $F(z)$ be a rational function of $z$ which is regular at $z = 0$ and so possesses a convergent power series

$$F(z) = \sum_{h=0}^{\infty} f_h z^h.$$

The problem arises of characterizing those rational functions $F(z)$ that have infinitely many vanishing Taylor coefficients $f_h$. After earlier and more special results by Siegel (2) and Ward (4) I applied in 1934 (1) a $p$-adic method due to Skolem (3) to this problem and obtained the following partial solution.

**Theorem 1.** Assume that all Taylor coefficients $f_h$ of the rational function $F(z)$ are algebraic numbers, and that infinitely many of them vanish. Then two integers $L$ and $L_1$ with $0 \leq L_1 < L$ exist such that $f_h$ is zero for all sufficiently large $h \equiv L_1 \pmod{L}$.

In the present paper, the restriction on the character of the coefficients $f_h$ will be removed, by showing the

**Theorem 2.** Theorem 1 remains valid when the coefficients $f_h$ of $F(z)$ are arbitrary complex numbers.

In the proof of this theorem, the assertion will be reduced to one relating to rational functions with algebraic Taylor coefficients, and it will be assumed that the truth of Theorem 1 has already been established.

1. If the difference of two functions is a polynomial, all but finitely many of their Taylor coefficients are the same. Also to a given rational function one can always add a unique polynomial such that the sum function vanishes at the point at infinity.

Hence, without loss of generality, we shall assume from now on that the rational function $F(z)$ is not only regular at $z = 0$, but also it vanishes at $z = \infty$. We then call $F(z)$ a normed function. The restriction to normed functions considerably shortens the discussion.

2. Let $L$ and $L_1$ be two integers such that $0 \leq L_1 < L$. We say that $F(z)$ has the zero sequence $L_1 \pmod{L}$ if all but finitely many of the Taylor coefficients $f_h$ with $h \equiv L_1 \pmod{L}$ are zero.

This property may also be expressed in another form. Put

$$e = e^{2\pi i/L} \quad \text{and} \quad E(z) = \sum_{j=0}^{L-1} e^{jL_1} F(e^{-j}z).$$
Evidently
\[ E(z) = \sum_{h=0}^{\infty} \sum_{j=0}^{L-1} e^{i(L_1-h)} E(z) = L \sum_{h=0}^{\infty} \sum_{j=h \equiv L_1 \pmod{L}} e^{iL_1 z^h}, \]
and so \( E(z) \) reduces to a polynomial if \( L_1 \pmod{L} \) is a zero sequence of \( F(z) \). On the other hand, as \( F(z) \) is normed, all terms \( e^{iL_1} F(\varepsilon^{-i} z) \) of \( E(z) \) vanish at \( z = \infty \). The same is then true for \( E(z) \) itself, and so \( E(z) \) vanishes identically. Hence the stronger property
\[ f_h = 0 \text{ for all suffixes } h \equiv L_1 \pmod{L} \]
holds.

3. Assume again that \( L_1 \pmod{L} \) is a zero sequence of \( F(z) \). Let further \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be the distinct poles of \( F(z) \); by hypothesis, none of these poles lies at \( z = 0 \). Then \( F(\varepsilon^{-i} z) \) has the poles
\[ e^{i} \alpha_1, e^{i} \alpha_2, \ldots, e^{i} \alpha_n. \]
As was shown in § 2,
\[ E(z) = \sum_{j=0}^{L-1} e^{iL_1} F(\varepsilon^{-i} z) = 0. \]
Hence the poles of \( F(z) \) are cancelled by the poles of the \( L - 1 \) other functions \( e^{iL_1} F(\varepsilon^{-i} z) \) where \( j = 1, 2, \ldots, L - 1 \).

It follows therefore that to every pole \( \alpha_\nu \) of \( F(z) \) there is a second pole \( \alpha_\mu \) \( (\mu + \nu \neq 0) \) such that \( \alpha_\nu / \alpha_\mu + 1 = 1 \) is an \( L \)-th root of unity, which, of course, need not be primitive. Furthermore, \( F(z) \) has at least two distinct poles.

4. Let \( \Sigma = \{ \alpha_\nu / \alpha_\mu \} \) be the set of all those quotients \( \alpha_\nu / \alpha_\mu + 1 \) of distinct poles of \( F(z) \) that are roots of unity. Unless \( \Sigma \) is the null set, there exists a smallest positive integer \( M \) such that \( \Sigma \) consists only of \( M \)-th roots of unity which, however, need not all be primitive.

Assume, in particular, that \( L_1 \pmod{L} \) is a zero sequence of \( F(z) \), and put
\[ (L, M) = L^*, \quad L' = \frac{L}{L^*}, \]
so that
\[ L^* = L\Lambda + MM, \quad L = L^*L', \]
with certain integers \( \Lambda \) and \( M \). By § 3, \( \Sigma \) is now certainly not the null set, because it contains elements that are \( L \)-th roots of unity. Denote by \( \Sigma^* \) the subset formed by all these elements of \( \Sigma \) that are \( L \)-th roots of unity. Thus the elements \( \alpha_\nu / \alpha_\mu \) of \( \Sigma^* \) satisfy both equations
\[ \left( \frac{\alpha_\nu}{\alpha_\mu} \right)^L = 1 \text{ and } \left( \frac{\alpha_\nu}{\alpha_\mu} \right)^M = 1, \]
and so also the equation
\[ \left( \frac{\alpha_\nu}{\alpha_\mu} \right)^{L^*} = \left( \left( \frac{\alpha_\nu}{\alpha_\mu} \right)^L \right)^\Lambda \left( \left( \frac{\alpha_\nu}{\alpha_\mu} \right)^M \right)^M = 1. \]
Therefore \( \Sigma^* \) consists only of \( L^* \)-th roots of unity.

5. We introduce now the \( L' \) new functions
\[ E_k(x) = \sum_{j=0}^{L-1} e^{iL_1} F(\varepsilon^{-i} z) \quad (k = 0, 1, 2, \ldots, L' - 1). \]
As already shown, the sum of these functions

\[ E(z) = \sum_{k=0}^{L'-1} E_k(z) = \sum_{j=0}^{L-1} e^{jLz}F(e^{-j}z) \]

is identically zero.

It is obvious that \( E_0(z) \) may have poles only at the points \( e^{j\pi_k}, e^{j\pi_2}, \ldots, e^{j\pi_n} \), where \( j \equiv 0 \pmod{L'} \), \( 0 \leq j \leq L - 1 \), while, for \( k = 1, 2, \ldots, L' - 1 \), poles of \( E_k(z) \) can lie only at \( e^{i\pi_k}, e^{i\pi_2}, \ldots, e^{i\pi_n} \), where \( i \equiv k \pmod{L'} \), \( 0 \leq i \leq L - 1 \). Let us suppose that \( e^{i\pi_v} \) is a pole of \( E_0(z) \), and that \( e^{i\pi_u} \) is one of \( E_k(z) \), where \( 1 \leq k \leq L' - 1 \). Then

\[ t - j \equiv k \neq 0 \pmod{L'}. \]

Hence

\[ L^*(t - j) \not\equiv 0 \pmod{L}, \]

whence

\[ (e^{i\pi_j})^{L^*} = 1, \quad e^{i\pi_j} \not\equiv 1. \]

Therefore, necessarily,

\[ e^{i\pi_v} = e^{i\pi_u}, \]

because, \( e^{i\pi_j} \) being an \( L^* \)th root of unity, the quotient

\[ \frac{\pi_v}{\pi_u} = e^{i\pi_j} \equiv 1 \]

would otherwise belong to \( \Sigma^* \) and be an \( L^* \)th root of unity; and this is not the case.

The function \( E_0(z) \) has then no poles in common with the other terms \( E_k(z) \) of \( E(z) \), and all its poles are also poles of \( E(z) \). Since \( E(z) \) has no poles, \( E_0(z) \) is thus a polynomial. But, from its definition in terms of \( F(z) \), \( E_0(z) \) is a normed rational function. Hence, finally,

\[ E_0(z) \text{ is identically zero.} \]

Put now

\[ \eta = e^{L'} = e^{2\pi i L^*}. \]

Evidently

\[ E_0(z) = \sum_{j=0}^{L^*-1} \eta^{jLz}F(\eta^{-j}z) = L^* \sum_{h=0}^{\infty} f_hz^h \equiv 0, \quad h \equiv L_1 \pmod{L^*} \]

whence

\[ f_h = 0 \quad \text{for all suffixes} \quad h \equiv L_1 \pmod{L^*}. \]

The following result has thus been established.

**Lemma 1.** Let \( L_1 \pmod{L} \) be a zero sequence of \( F(z) \); let \( \pi_1, \pi_2, \ldots, \pi_n \) be the distinct poles of \( F(z) \); and let \( M \) be the smallest integer such that all quotients \( \frac{\pi_v}{\pi_u} = 1 \), that are roots of unity, are \( M \)-th roots of unity. If \( L^* = (L, M) \), then \( F(z) \) admits the zero sequence \( L_1 \pmod{L^*} \).

This lemma is of importance for later, because \( L^* \) is a divisor of \( M \), and \( M \) depends only on the poles of \( F(z) \). We note that the lemma remains valid when \( F(z) \) is not normed, but shall not use this fact.

6. We proceed now to the proof of Theorem 2.

The most general rational function \( F(z) \not\equiv 0 \) regular at \( z = 0 \) is of the form

\[ F(z) = \frac{a_0 + a_1z + \ldots + a_mz^m}{(z - \pi_1)^{l_1}(z - \pi_2)^{l_2} \ldots (z - \pi_n)^{l_n}}. \]
Here $e_1, e_2, \ldots, e_n$ are arbitrary positive integers; $a_0, a_1, \ldots, a_m$ are arbitrary complex numbers with $a_m \neq 0$; and $x_1, x_2, \ldots, x_n$, the poles of $F(z)$, are complex numbers that are all distinct and different from zero, but are otherwise arbitrary. The function $F(z)$ is assumed to be normed, and therefore the inequality

$$m < e_1 + e_2 + \ldots + e_n$$

holds.

If again

$$F(z) = \sum_{h=0}^{\infty} f_hz^h$$

is the power series of $F(z)$ in the neighbourhood of $z = 0$, let $H$ be the set of all suffixes $h$ for which

$$f_h = 0.$$

It is assumed that $H$ is an infinite set; the problem is to prove that under this hypothesis $F(z)$ possesses at least one zero sequence.

7. From now on let $X$ be the set of the $m + n + 1$ parameters

$$X = \{a_0, a_1, \ldots, a_m, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$$

that occur in $F(z)$; the use of $x_\nu^{-1}$ rather than $x_\nu$ will prove to be an advantage. Further put

$$e_0 = e_1 + e_2 + \ldots + e_n.$$

Then $F(z)$ may also be written in the form

$$F(z) = (-1)^{e_0} \prod_{\nu=1}^{n} (x_\nu^{-1})^{e_\nu} \sum_{\mu=0}^{m} a_\mu z^\mu \prod_{\nu=1}^{n} (1 - a_\nu^{-1}z)^{-e_\nu}.$$

On developing here the last factor into a power series by means of the binomial theorem, we see immediately that, for $h = 0, 1, 2, \ldots, f_h$ is a polynomial with rational coefficients in the elements of $X$.

Hence, if $X$ consists only of algebraic numbers, the coefficients $f_h$ are likewise algebraic. It is assumed that this is no longer the case; hence $X$ includes at least one transcendental number.

Denote by $R$ the Gaussian imaginary quadratic field. The elements of $X$ generate a smallest extension field

$$\mathbb{P} = R(X) = R(a_0, a_1, \ldots, a_m, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$$

over $R$. It is shown in algebra that this extension field may be constructed as follows.

We first adjoin to $R$ a certain finite set of transcendental complex numbers

$$\sigma_1, \sigma_2, \ldots, \sigma_\rho$$

that are algebraically independent over $R$, so arriving at the purely transcendental extension

$$\mathbb{P}_0 = R(\sigma_1, \sigma_2, \ldots, \sigma_\rho).$$

The field $\mathbb{P}$ is now derived from $\mathbb{P}_0$ by a simple algebraic extension

$$\mathbb{P} = \mathbb{P}_0(\tau) = R(\sigma_1, \sigma_2, \ldots, \sigma_\rho, \tau),$$

$\tau$ being a suitable complex number algebraic over $\mathbb{P}_0$. 
Taylor coefficients of rational functions

This number $\tau$ may still be chosen in many different ways, and there is no loss of generality in assuming that $\tau$ is integral over the polynomial ring $R[\sigma_1, \sigma_2, \ldots, \sigma_p]$. The equation for $\tau$ has then the form

$$Q(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau) \equiv \tau^q + \sum_{k=1}^q Q_{\kappa}(\sigma_1, \sigma_2, \ldots, \sigma_p) \tau^{q-k} = 0,$$

where

$$Q_{\kappa}(\sigma_1, \sigma_2, \ldots, \sigma_p) \quad (\kappa = 1, 2, \ldots, q)$$

are polynomials in $R[\sigma_1, \sigma_2, \ldots, \sigma_p]$. It may also be assumed that $Q(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$, considered as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_p, \tau$, is irreducible over $R$.

8. The elements $a_\mu$ and $a_\nu^{-1}$ of $X$ are finite numbers in $P$. They can therefore be written as polynomials in $\tau$, with coefficients that are rational functions of $\sigma_1, \sigma_2, \ldots, \sigma_p$ with numerical coefficients in $R$. Denote by $\Delta(\sigma_1, \sigma_2, \ldots, \sigma_p) \neq 0$ the least common denominator of these rational functions; $\Delta$ is thus an element of $R[\sigma_1, \sigma_2, \ldots, \sigma_p]$. Then $a_\mu$ and $a_\nu^{-1}$ take the form

$$a_\mu = \frac{A_\mu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)}{\Delta(\sigma_1, \sigma_2, \ldots, \sigma_p)} \quad (\mu = 0, 1, \ldots, m)$$

and

$$a_\nu^{-1} = \frac{A_\nu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)}{\Delta(\sigma_1, \sigma_2, \ldots, \sigma_p)} \quad (\nu = 1, 2, \ldots, n).$$

Here the numerators

$$A_\mu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau), \quad A_\nu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$$

belong to the polynomial ring $R[\sigma_1, \sigma_2, \ldots, \sigma_p, \tau]$.

On substituting these expressions for the elements of $X$, $F(z)$ becomes a rational function

$$F(z) = \Phi(z \mid \sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$$

not only of $z$, but also of $\sigma_1, \sigma_2, \ldots, \sigma_p, \tau$, while its numerical coefficients lie in $R$. It follows further, from the representation of $f_h$ as a polynomial in the elements of $X$ with coefficients in $R$, that these Taylor coefficients may be written as

$$f_h(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau) = 0 \quad (h = 0, 1, 2, \ldots),$$

where the numerators

$$\phi_h(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$$

lie in the polynomial ring $R[\sigma_1, \sigma_2, \ldots, \sigma_p, \tau]$, while the exponents $d_h$ are certain positive integers depending on $h$. One may, in fact, choose $d_h = e_0 + h + 1$; but we shall not need this. The hypothesis on $f_h$ implies that

$$\phi_h(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau) = 0 \quad \text{if} \quad h \in H.$$

9. Let us now replace the algebraically independent complex numbers $\sigma_1, \sigma_2, \ldots, \sigma_p$ by independent complex variables

$$s_1, s_2, \ldots, s_p,$$

and the complex number $\tau$ for which

$$Q(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau) = 0$$
by a dependent complex variable $t$ satisfying
\[ Q(s_1, s_2, \ldots, s_p; t) = 0. \]

We then obtain a new rational function
\[ F^*(z) = \Phi(z \mid s_1, s_2, \ldots, s_p; t) \]
of $z$, as well as of $s_1, s_2, \ldots, s_p, t$, with numerical coefficients in $R$. This function has the explicit form
\[ F^*(z) = \frac{a_{s_1}^* z + \ldots + a_m^* z^m}{(z - \alpha_1^*)^c_1 (z - \alpha_2^*)^c_2 \ldots (z - \alpha_n^*)^c_n}, \]
where
\[ a_{s_1}^* = \frac{A_1(s_1, s_2, \ldots, s_p; t)}{\Delta(s_1, s_2, \ldots, s_p)} \quad (\mu = 0, 1, \ldots, m) \]
and
\[ \alpha_\nu^* = \frac{A_\nu(s_1, s_2, \ldots, s_p; t)}{\Delta(s_1, s_2, \ldots, s_p)} \quad (\nu = 1, 2, \ldots, n). \]

Further it possesses the power series
\[ F^*(z) = \sum_{h=0}^{\infty} f_h^* z^h, \]
where
\[ f_h^* = \frac{\Phi_h(s_1, s_2, \ldots, s_p; t)}{\Delta(s_1, s_2, \ldots, s_p)^{dh}} \quad (h = 0, 1, 2, \ldots). \]

Since $\Delta$ does not vanish identically, and since the change from $\sigma_1, \sigma_2, \ldots, \sigma_p, \tau$ to $s_1, s_2, \ldots, s_p, t$ maps $P = R(\sigma_1, \sigma_2, \ldots, \sigma_p, \tau)$ isomorphically onto $R(s_1, s_2, \ldots, s_p, t)$, it is clear that also
\[ \phi_h(s_1, s_2, \ldots, s_p, t) = 0 \quad \text{and} \quad f_h^* = 0 \quad \text{if} \quad h \in H. \]

It is further obvious from the construction that for
\[ s_1 = \sigma_1, \quad s_2 = \sigma_2, \quad \ldots, \quad s_p = \sigma_p, \quad t = \tau, \]
the equations
\[ F^*(z) = F(z), \quad a_{s_1}^* = a_{\sigma_1}, \quad \alpha_\nu^* = \alpha_\nu, \quad f_h^* = f_h \]
hold.

10. To simplify the notation, we introduce the $p$-dimensional space $C^p$ of all points
\[ s = (s_1, s_2, \ldots, s_p), \quad s' = (s'_1, s'_2, \ldots, s'_p), \quad \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p), \quad \ldots, \]
with arbitrary real or complex coordinates, and we make $C^p$ a metric space by defining the distance $\rho(s, s')$ of two points $s, s'$ by
\[ \rho(s, s') = \sqrt{\left| s_1 - s'_1 \right|^2 + \left| s_2 - s'_2 \right|^2 + \ldots + \left| s_p - s'_p \right|^2}. \]

Let $R^p$ similarly be the set of all points in $C^p$ the coordinates of which lie in the Gaussian field $R$; thus $R^p$ is dense in $C^p$. We can then select in many ways an infinite sequence of points
\[ S = \{s^{(1)}, s^{(2)}, s^{(3)}, \ldots\}, \quad \text{where} \quad s^{(k)} = (s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}), \]
in $R^p$ such that
\[ \lim_{k \to \infty} s^{(k)} = \sigma, \quad \text{i.e.} \quad \lim_{k \to \infty} \rho(s^{(k)}, \sigma) = 0. \]

From the form of the equation
\[ Q(s_1, s_2, \ldots, s_p, t) = 0 \]
for \( t \), it is further possible to associate with each point \( s^{(k)} \) of \( S \) a complex root, \( t^{(k)} \) say, of the equation
\[
Q(s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}, t^{(k)}) = 0,
\]
such that also
\[
\lim_{k \to \infty} t^{(k)} = \tau.
\]

11. Denote, for \( k = 1, 2, 3, \ldots \), by
\[
F^{(k)}(z), \quad a^{(k)}_\mu, \quad \alpha^{(k)}_\nu, \quad f^{(k)}_h,
\]
the expressions into which
\[
F^*(z), \quad a^*_\mu, \quad \alpha^*_\nu, \quad f^*_h,
\]
respectively, are changed on putting
\[
s_1 = s_1^{(k)}, \quad s_2 = s_2^{(k)}, \quad \ldots, \quad s_p = s_p^{(k)}, \quad t = t^{(k)}.
\]
Then \( F^{(k)}(z) \) is the rational function
\[
F^{(k)}(z) = \frac{a_0^{(k)} + a_1^{(k)}z + \ldots + a_m^{(k)}z^m}{(z - \alpha_1^{(k)})^\epsilon_1(z - \alpha_2^{(k)})^\epsilon_2\ldots(z - \alpha_n^{(k)})^\epsilon_n} = \Phi(z \mid s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}, t^{(k)})
\]
of \( z \) with the Taylor series
\[
F^{(k)}(z) = \sum_{h=0}^{\infty} f^{(k)}_hz^h,
\]
and here
\[
f^{(k)}_h = 0 \quad \text{if} \quad h \in H.
\]
We must, however, assume that \( k \) is already sufficiently large, i.e. that \( s^{(k)} \) is sufficiently near to \( \sigma \), so as to exclude the possibility that one of the expressions \( a^{(k)}_\mu, \alpha^{(k)}_\nu, f^{(k)}_h \) becomes infinite, or that one of the poles \( \alpha^{(k)}_\nu \) vanishes, or that two of these poles coincide. Assume, say, that these cases are excluded when \( k > k_0 \).

It follows now, from the continuity properties of a rational function, that
\[
\lim_{k \to \infty} a^{(k)}_\mu = a_\mu, \quad \lim_{k \to \infty} \alpha^{(k)}_\nu = \alpha_\nu, \quad \lim_{k \to \infty} f^{(k)}_h = f_h
\]
for all values of the suffixes \( \mu, \nu \) and \( h \).

12. The equation
\[
Q(s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}, t^{(k)}) = 0
\]
for \( t^{(k)} \) is of degree \( q \), and its coefficients lie in \( R \); for both the numerical coefficients of \( Q \), and the coordinates of \( s^{(k)} \), belong to \( R \). Therefore \( t^{(k)} \) is an algebraic number at most of degree \( q \) over the Gaussian field, hence at most of degree \( 2q \) over the rational field. Denote by \( K^{(k)} = R(t^{(k)}) \) the algebraic extension of \( R \) generated by \( t^{(k)} \); this field has likewise a degree not greater than \( 2q \) over the rational field. From their definitions, it is clear that the numbers
\[
a^{(k)}_\mu, \quad \alpha^{(k)}_\nu, \quad f^{(k)}_h
\]
all are elements of \( K^{(k)} \), as soon as \( k > k_0 \).

In particular, the Taylor coefficients \( f^*_h \) of
\[
F^*(z) = \sum_{h=0}^{\infty} f^*_hz^h
\]
are algebraic numbers, and furthermore infinitely many of these coefficients vanish,
\[
f^*_h = 0 \quad \text{if} \quad h \in H.
\]
The hypothesis of Theorem 1 is thus satisfied. Hence, for every \( k \geq k_0 \), \( F^{(k)}(z) \) possesses at least one zero sequence \( L_1 \) (mod \( L \)). Here we may assume that \( 0 \leq L_1 < L \). Both \( L = L^{(k)} \) and \( L_1 = L_1^{(k)} \) may still depend on \( k \). We note that, by hypothesis,

\[
m < e_1 + e_2 + \ldots + e_n.
\]

Hence also \( F^{(k)}(z) \) is normed, so that all its Taylor coefficients \( f^{(k)}_h \) satisfying \( h \equiv L_1 \) (mod \( L \)) are zero.

13. Lemma 1 enables us to construct a zero sequence \( L_1 \) (mod \( L \)) of \( F^{(k)}(z) \) with bounded \( L \), hence also with bounded \( L_1 \).

The poles \( \alpha^{(k)}_1, \alpha^{(k)}_2, \ldots, \alpha^{(k)}_n \) of \( F^{(k)}(z) \) lie in \( K^{(k)} \), and the same is therefore true for the quotients of two such poles. Denote by \( \Sigma = \Sigma^{(k)} \) the set of all those quotients

\[
\frac{\alpha^{(k)}_\mu}{\alpha^{(k)}_\nu} \neq 1
\]

that are roots of unity; we know, from § 3, that \( \Sigma \) is not the null set. Hence a smallest positive integer \( M = M^{(k)} \) exists such that all elements of \( \Sigma \) are \( M \)th roots of unity.

By Lemma 1, \( F^{(k)}(z) \) admits also the larger zero sequence

\[ L_1 \) (mod \( L^* \), where \( L^* = (L, M). \)

This zero sequence is identical with \( L_1^* \) (mod \( L^* \)), where \( L_1^* \) is the integer for which

\[ L_1^* \equiv L_1 \) (mod \( L^* \), \( 0 \leq L_1^* < L^* \).

The roots of unity which are the elements of \( \Sigma \) lie in the algebraic field \( K^{(k)} \), and this field is at most of degree \( 2q \). On the other hand, there are only finitely many roots of unity that are algebraic numbers at most of degree \( 2q \). Denote by \( M_0 \) the least common multiple of the orders of all these roots of unity. Then evidently

\[ M^{(k)} \leq M_0 \) for \( k \geq k_0. \]

Since \( L^* \) is a divisor of \( M^{(k)} \), this implies that also

\[ 0 \leq L_1^* < L^* \leq M_0 \) for \( k \geq k_0. \]

14. On dropping now again the asterisk, the last result may be formulated as follows:

If \( k \geq k_0 \), then \( F^{(k)}(z) \) possesses at least one zero sequence

\[ L_1 \) (mod \( L \), where \( 0 \leq L_1 < L \leq M_0, \)

and where \( M_0 \) is independent of \( k \). Moreover, all Taylor coefficients \( f^{(k)}_h \) of \( F^{(k)}(z) \) with \( h \equiv L_1 \) (mod \( L \)) are zero.

There exist only finitely many zero sequences \( L_1 \) (mod \( L \)) for which \( 0 \leq L_1 < L \leq M_0 \), the zero sequences

\[ Z_1, Z_2, \ldots, Z_v, \]
say. For each \( k \geq k_0 \) denote by \( u = u^{(k)} \) the smallest suffix such that \( Z_u \) is a zero sequence of \( F^{(k)}(z) \). This function \( u = u^{(k)} \) has only \( v \) possible values. Hence there is an infinite sequence of indices

\[ k = k_1, k_2, k_3, \ldots, \) where \( k_0 \leq k_1 \leq k_2 \leq k_3 \leq \ldots, \]
for which \( u = u^{(k)} \) assumes one and the same fixed value \( u_0 \). For all these indices \( F^{(k)}(z) \) possesses the same zero sequence \( Z_{u_0} \), or say \( L_1^0 \) (mod \( L^0 \)), and all Taylor coefficients \( f^{(k)}_h \) with \( h \equiv L_1^0 \) (mod \( L^0 \)) are zero. However, as was proved in §11,

\[
\lim_{k \to \infty} f^{(k)}_h = f_h \quad \text{for all } h.
\]

Therefore, on allowing \( k \) to run over the sequence \( k_1, k_2, k_3, \ldots \) to infinity, it follows at once that also

\[
f_h = 0 \quad \text{if} \quad h \equiv L_1^0 \quad \text{(mod } L^0)\).
\]

Hence the original function \( F(z) \) likewise admits the zero sequence \( L_1^0 \) (mod \( L^0 \)). This proves the assertion.

15. Theorem 2 implies a slightly stronger result.

**Theorem 3.** Let \( F(z) = \sum_{h=0}^{\infty} f_h z^h \) be a rational function of \( z \) which is regular at \( z = 0 \) and has infinitely many vanishing Taylor coefficients \( f_h \). Then a positive integer \( L \) and at most \( L \) non-negative integers \( L_1, L_2, \ldots, L_l \) exist such that \( f_h \) vanishes exactly when

\[
h \equiv L_j \quad \text{(mod } L) , \quad h \geq L_j \quad (j = 1, 2, \ldots, l)
\]

and for at most finitely many other values of \( h \).

**Proof.** It may again be assumed that \( F(z) \) is normed. Denote by \( M \) the same positive integer as in Lemma 1. By this lemma, it suffices to consider those zero sequences \( L_j \) (mod \( L \)) of \( F(z) \) for which \( L \) is a divisor of \( M \). As such sequences can be subdivided into sequences (mod \( M \)), it further suffices to prove the theorem with \( L \) replaced by \( M \).

Denote by \( L_1, L_2, \ldots, L_l \) (mod \( M \)) all distinct residue classes \( h \equiv L_j \) (mod \( M \)) that contain infinitely many suffixes \( h \) for which \( f_h = 0 \). The assertion is proved if it can be shown that each \( L_j \) (mod \( M \)) is a zero sequence of \( F(z) \). It will be enough to consider \( L_1 \) (mod \( M \)).

We assume thus that

\[
f_h = 0 \quad \text{for infinitely many } h \equiv L_1 \quad \text{(mod } M) .
\]

Similarly as before, put

\[
e = e^{2\pi i/M}, \quad E(z) = \sum_{j=0}^{M-1} e^{L_j z} F(e^{-jz});
\]

further write

\[
\zeta = z^M.
\]

Then

\[
z^{-L_1} E(z) = M z^{-L_1} \sum_{h=0}^{\infty} f_h z^h = M \sum_{k=0}^{\infty} f_{L_1+kM} z^{kM} = E(\zeta),
\]

where

\[
E(\zeta) = M \sum_{k=0}^{\infty} f_{L_1+kM} \zeta^k
\]
evidently is a rational function of \( \zeta \). This new function \( E(\zeta) \) is regular at \( \zeta = 0 \), vanishes at \( \zeta = \infty \), and has infinitely many vanishing Taylor coefficients \( f_{L_1+kM} \).
Hence it follows from Theorem 2 that \( E(\xi) \) possesses at least one zero sequence

\[ k \equiv \kappa_1 \pmod{\kappa} \]

As the function is normed, this implies that

\[ f_{L_1 + \kappa M} = 0 \quad \text{if} \quad k \equiv \kappa_1 \pmod{\kappa}, \]

or, what is the same,

\[ f_h = 0 \quad \text{if} \quad h \equiv L_1 + \kappa_1 M \pmod{\kappa M}. \]

This relation means that the original function \( F(z) \) has the zero sequence \( L_1 + \kappa_1 M \pmod{\kappa M} \). But then, by Lemma 1, it also admits the larger zero sequence \( L_1 + \kappa_1 M \pmod{M} \), hence also the zero sequence \( L_1 \pmod{M} \). This concludes the proof.

16. It is well known that, for sufficiently large \( h \), the Taylor coefficient \( f_h \) of the rational function \( F(z) \) has the explicit representation

\[ f_h = \sum_{\nu=1}^{n} p_{\nu}(h) \beta_{\nu}^h \]

where \( p_1(h), p_2(h), \ldots, p_n(h) \) are polynomials in the variable \( h \) not identically zero, while \( \beta_1, \beta_2, \ldots, \beta_n \) are distinct constants different from zero, viz. the reciprocals of the poles of \( F(z) \). Conversely, every expression of this kind defines the Taylor coefficients of a rational function regular at \( z = 0 \), and the same is true if \( h \) is replaced by \( -h \).

The following result is then implicit in Theorem 2.

**Theorem 4.** Let \( \beta_1, \beta_2, \ldots, \beta_n \) be finitely many complex numbers that are distinct, different from zero, and such that no quotient of two of them is a root of unity. Also let \( p_1(h), p_2(h), \ldots, p_n(h) \) be an equal number of polynomials not identically zero with arbitrary complex coefficients. Then the equation

\[ \sum_{\nu=1}^{n} p_{\nu}(h) \beta_{\nu}^h = 0 \]

has at most finitely many solutions in rational integers.

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**REFERENCES**


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