AN INEQUALITY FOR A PAIR OF POLYNOMIALS THAT ARE RELATIVELY PRIME

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To T. M. Cherry

Let \( f(x) \) and \( g(x) \) be two polynomials with arbitrary complex coefficients that are relatively prime. Hence the maximum

\[
m(x) = \max (|f(x)|, |g(x)|)
\]

is positive for all complex \( x \). Since \( m(x) \) is continuous and tends to infinity with \( |x| \), the quantity

\[
E(f, g) = \min_x m(x)
\]

is therefore also positive.

In the theory of transcendental numbers one often requires a good positive estimate for \( E(f, g) \). The usual method for obtaining such an estimate is as follows. If \( R(f, g) \) denotes the resultant of \( f \) and \( g \), then identically in \( x \)

\[
f(x)F(x) + g(x)G(x) = R(f, g)
\]

where \( F(x) \) and \( G(x) \) are two polynomials that can be defined explicitly in terms of determinants. It follows that

\[
m(x) \geq |R(f, g)|/\{|F(x)| + |G(x)|\},
\]

and hence it suffices to give an upper estimate for \( |F(x)| + |G(x)| \). For this purpose one may assume that \( |x| \) is not too large; for when \( |x| \) is large, \( m(x) \) trivially cannot be small. (See e.g. A. O. Gelfond, Transcendentnye i algebritcheskie tchisla, Moskva 1952, pp. 181–2.)

In the present note I shall apply a different and better method that is due to N. Feldman. It has the additional advantage of leading to a best-possible result.

1. Let, in explicit form,

\[
f(x) = a_0(x-\alpha_1) \cdots (x-\alpha_m), \quad g(x) = b_0(x-\beta_1) \cdots (x-\beta_n),
\]

where \( a_0 \neq 0 \) and \( b_0 \neq 0 \), and where \( \alpha_h \neq \beta_k \) for all \( h \) and \( k \). Put, for any given complex number \( x \),

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\[ \alpha = \min_{1 \leq h \leq m} |x - \alpha_h| \quad \text{and} \quad \beta = \min_{1 \leq k \leq n} |x - \beta_k|, \]

and denote by \( r \) and \( s \) two suffixes for which

\[ \alpha = |x - \alpha_r| \quad \text{and} \quad \beta = |x - \beta_s|. \]

Then at least one of the two numbers \( \alpha \) and \( \beta \) is positive. Assume, first, that

\[ 0 < \alpha \leq \beta \]

and number the zeros of \( g(x) \) such that, say,

\[ |\alpha_r - \beta_k| \begin{cases} < 2\alpha & \text{if } k = 1, 2, \cdots, N, \\ \geq 2\alpha & \text{if } k = N+1, N+2, \cdots, n; \end{cases} \]

here \( N \) is a certain integer satisfying \( 0 \leq N \leq n \).

If \( k = 1, 2, \cdots, N \), then

(1) \[ |x - \beta_k| \geq \beta \geq \alpha > |\alpha_r - \beta_k|/2. \]

If, however, \( k = N+1, N+2, \cdots, n \), then

\[ |\alpha_r - \beta_k| \geq 2\alpha = 2|x - \alpha_r| \]

and therefore

(2) \[ |x - \beta_k| = |(x - \alpha_r) + (\alpha_r - \beta_k)| \geq |\alpha_r - \beta_k| - |x - \alpha_r| \geq |\alpha_r - \beta_k|/2. \]

On combining the inequalities (1) and (2) it follows that

\[ |g(x)| = |b_0 \prod_{k=1}^{n} (x - \beta_k)| \geq 2^{-n} |b_0 \prod_{k=1}^{n} (\alpha_r - \beta_k)|. \]

It is obvious that this formula remains true also when

\[ \alpha = 0, \]

and hence we have proved that

\[ |g(x)| \geq 2^{-n} |g(\alpha_r)| \quad \text{if} \quad 0 \leq \alpha \leq \beta. \]

In exactly the same way it follows that

\[ |f(x)| \geq 2^{-m} |f(\beta_s)| \quad \text{if} \quad 0 \leq \beta \leq \alpha. \]

These two inequalities together imply the following result.

**Theorem 1.** Let \( f(x) \) and \( g(x) \) have the degrees \( m \) and \( n \) and the zeros \( \alpha_1, \cdots, \alpha_m \) and \( \beta_1, \cdots, \beta_n \), respectively. Then

\[ E(f, g) \geq \min_{1 \leq h \leq m} (2^{-m} |f(\beta_k)|, 2^{-n} |g(\alpha_h)|). \]
This result is best possible because the assertion holds with equality in the special case when \( f \) and \( g \) are the two polynomials

\[
f(x) = (x-1)^m \quad \text{and} \quad g(x) = (x+1)^n.
\]

2. Theorem 1 gives a lower bound for \( E(f, g) \) in terms of the zeros of \( f \) and \( g \). It is now not difficult to replace this estimate by one that involves instead only the coefficients of these two polynomials.

Let in explicit form

\[
f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \quad g(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n.
\]

Further denote by

\[
L(f) = |a_0| + |a_1| + \cdots + |a_m|, \quad L(g) = |b_0| + |b_1| + \cdots + |b_n|
\]

the lengths of the two polynomials. By a theorem of R. Güting, \(^1\)

\[|f(\beta_k)| \gtrsim |R(f, g)|/L(f)^{n-1}L(g)^m, \quad |g(\alpha_h)| \gtrsim |R(f, g)|/L(f)^{n}L(g)^{m-1}\]

for all suffixes \( h \) and \( k \). Hence, by Theorem 1,

\[E(f, g) \gtrsim |R(f, g)|L(f)^{-n}L(g)^{-m} \min \{2^{-m}L(f), 2^{-n}L(g)\}.
\]

For the applications, the most important case is that of polynomials with integral coefficients. The resultant \( R(f, g) \neq 0 \) is then also an integer and hence its absolute value is not less than 1. Therefore, in this particular case,

\[E(f, g) \gtrsim L(f)^{-n}L(g)^{-m} \min \{2^{-m}L(f), 2^{-n}L(g)\}.
\]

While this formula is very simple, it is, however, no longer best possible.

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\(^1\) Approximation of algebraic numbers by algebraic numbers, Michigan Math. J. 8 (1961), 149—159.