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from the
MICHIGAN MATHEMATICAL JOURNAL
vol. 12 (1965)
pp. 417-420
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Introduction. Equations in free groups have recently attracted considerable attention (see, for example, R. C. Lyndon and M. P. Schützenberger [3], G. Baumslag [1]). Free metabelian groups share many properties with free groups, and we now prove an analogue of a theorem about equations in free groups.

THEOREM. If $a$ and $b$ are elements of a free metabelian group that are linearly independent modulo the derived group, and if $n$ is any integer greater than 1, then $a^n b^n$ is not an $n$-th power.

This theorem leaves unanswered a host of related questions. For example, if $\ell$, $m$, and $n$ are integers greater than 1, can $a^\ell b^m$ be an $n$-th power? This certainly seems unlikely. Of course, $a$ and $b$ must be linearly independent modulo the derived group; for if $u$ and $v$ are elements of a metabelian group and $v$ lies in the derived group, then

$$(u^{-1})^2 (uv^2)^2 = (u^{-1} vu \cdot v)^2.$$

We effect the proof of our theorem by first reducing it in a standard way to a problem in the group ring over the integers of a free abelian group (see G. Baumslag, Bernhard H. Neumann, Hanna Neumann, and Peter M. Neumann [2]) and then solving this problem with the help of elementary algebraic number theory.

The reduction to the group ring. Suppose that $a$ and $b$ are elements of a free metabelian group $M$ and that they are linearly independent modulo $M'$, the derived group of $M$. By a theorem of Nielsen [4] it follows that we can find an automorphism $\theta$ of $M$ and a free set of generators $x$, $y$, $z$, $\cdots$ such that

$$a^\theta \equiv x^\alpha (M'), \quad b^\theta \equiv y^\beta (M') \quad (\alpha > 0, \beta > 0).$$

We may therefore assume

$$a \equiv x^\alpha (M'), \quad b \equiv y^\beta (M') \quad (\alpha > 0, \beta > 0). \quad (1)$$

The homomorphism $\eta$ of $M$ into $M$ defined by

$$x\eta = x, \quad y\eta = y, \quad z\eta = 1, \quad \cdots$$

maps $M$ into a free metabelian group of rank 2 in which $a\eta$ and $b\eta$ are themselves linearly independent modulo the derived group. Thus it suffices to settle the theorem for a free metabelian group $M$ of rank 2 on $x$ and $y$ with $a$ and $b$ given by (1).

As usual, we put

Received March 23, 1965.
This paper was written at the Institute for Advanced Studies in Canberra. The first author gratefully acknowledges a grant from the National Science Foundation and the hospitality of the Australian National University.
\[(u^n_1)v_1(u^n_2)v_2 \ldots (u^n_m)v_m = u^{n_1}v_1 + n_2v_2 + \cdots + n_mv_m,\]

where \(u, v_1, \ldots, v_m\) are elements of \(M\) and \(n_1, \ldots, n_m\) are integers.

Now let \(k = x^{-1}y^{-1}xy\). It is well-known that then every element of \(M'\) can be uniquely represented in the form \(k^F(x, y)\), where \(F(x, y)\) is an element of the group ring \(R\) of the free abelian group \(M/M'\). Thus \(F(x, y)\) is a finite Laurent series of the form \(\sum \gamma_{i,j}x^i y^j\), where \(\gamma_{i,j}, i,\) and \(j\) are integers. It follows that every element of \(M\) can be written uniquely in the form \(x^\lambda y^\mu k^F\), where \(\lambda\) and \(\mu\) are integers and \(F\) is in \(R\).

Assume now that \(a^n b^n = c^n\), where \(a\) and \(b\) are given by (1); we may clearly assume \(n\) is a prime. Thus \(c = x^\alpha y^\beta (M')\). Therefore we have the relations

\[a = x^\alpha k^A, \quad b = y^\beta k^B, \quad c = x^\alpha y^\beta k^C \quad (A, B, C \in R).\]

If we abbreviate \(\frac{z^t - 1}{z - 1}\) to \(z^t - 1\), then it is easy to show that

\[a^n = x^{\alpha n} k^{\frac{A}{\alpha - 1}} ;\]

similarly for \(b^n\) and \(c^n\). Thus \(a^n b^n = c^n\) takes the form

\[\sum_{i=1}^{n-1} \gamma_{i,j} x^i y^j\]

Moreover, if \(u\) and \(v\) are elements of a metabelian group, then

\[uv^n = u^n v^{n-1} \equiv [u, v] \quad \text{for } i = 1, \ldots, n-1.\]

Now

\[[y^\beta, x^\alpha] = [x^\alpha, y^\beta]^{-1} = k^{\frac{x^\alpha y^\beta - 1}{y^\beta - 1}}.

Therefore it follows that

\[(x^\alpha y^\beta)^n = x^{\alpha n} y^{\beta n} k^D,\]

where

\[D = \left(\frac{x^\alpha - 1}{x - 1}\right) \left(\frac{y^\beta - 1}{y - 1}\right) \sum_{i=1}^{n-1} y^{\beta i} x^{\alpha(i-1)} \frac{y^{\beta(n-1)} - 1}{y^\beta - 1}.\]

We see then from (2) that in the group ring \(R\) we have the relation

\[A(1 + x^\alpha + \cdots + x^{\alpha(n-1)}) y^{\beta n} + B(1 + y^\beta + \cdots + y^{\beta(n-1)})\]

\[= D + C(1 + x^\alpha y^\beta + \cdots + (x^\alpha y^\beta)^{n-1}).\]
The analysis of (4). Let \( A_1(\alpha^x, y^\beta) \) be the sum of all terms \( \alpha_{i,j} x^i y^j \) in \( A \) in which \( i \) and \( j \) are multiples of \( \alpha \) and \( \beta \), respectively, and define \( B_1, C_1, D_1 \) similarly. If we now put \( X = x^\alpha \), \( Y = y^\beta \), then it follows from (3) and (4) that

\[
A_1(X, Y)(1 + X + \cdots + X^{n-1})Y^n + B_1(X, Y)(1 + Y + \cdots + Y^{n-1})
\]

\[
= D_1(X, Y) + C_1(X, Y)(1 + XY + \cdots + (XY)^{n-1}).
\]

Now, by (3),

\[
D_1(X, Y) = \sum_{i=1}^{n-1} Y^i X^{i-1} \left( \frac{Y^{n-i} - 1}{Y - 1} \right).
\]

Put \( X = z^{-1}, Y = z \) in (5), where \( z \) is a primitive \( n \)-th root of unity. Then (5) reduces to

\[0 = D_1(z^{-1}, z) + nC_1(z^{-1}, z).\]

Clearly, \( d = D_1(z^{-1}, z) \) and \( e = C_1(z^{-1}, z) \) are algebraic integers. However, by (6), we find that

\[d = \sum_{i=1}^{n-1} z \left( \frac{z^{n-i} - 1}{z - 1} \right) = z \left[ \frac{(z^{n-1} - 1) + \cdots + (z - 1) + (1 - 1)}{z - 1} \right] = \frac{nz}{z - 1}.
\]

This means that \(-e = \frac{z}{z - 1} = 1 + \frac{1}{z - 1}\). Hence

\[\frac{1}{z - 1} = -e - 1\]

is an algebraic integer. But \( z \), and therefore also \( w = z - 1 \), is an algebraic integer of degree \( n - 1 \). However, \((w + 1)^n - 1 = 0\). Since \( n > 1 \), \( w^n + nw^{n-1} + \cdots + nw = 0 \), and so also

\[w^{n-1} + nw^{n-2} + \cdots + n = 0.\]

This polynomial in \( w \) is therefore irreducible. Thus we find that \( w^{-1} \) is a root of an irreducible polynomial of the form

\[f = n\xi^{n-1} + \cdots + n\xi + 1.\]

Therefore \( w^{-1} \) is not an integer. This contradiction completes the proof of the theorem.
Added in proof. R. C. Lyndon has recently shown that for any three relatively prime integers \(\ell, m, n (\ell > 1, m > 1, n > 1)\) and every free metabelian group \(M\) of rank at least 2, there exist elements \(a, b, c\), with \(a\) and \(b\) independent modulo \(M'\), such that

\[
a^\ell b^m = c^n.
\]

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