AN UNSOLVED PROBLEM ON THE POWERS OF $\frac{3}{2}$

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Let $\alpha$ be an arbitrary positive number. For every integer $n \geq 0$ we can write

$$\alpha\left(\frac{3}{2}\right)^n = g_n + r_n,$$

where

$$g_n = \lfloor \alpha\left(\frac{3}{2}\right)^n \rfloor$$

is the largest integer not greater than $\alpha\left(\frac{3}{2}\right)^n$, i.e. the integral part of $\alpha\left(\frac{3}{2}\right)^n$, and $r_n$ is its fractional part and so satisfies the inequality

$$0 \leq r_n < 1.$$

We say that $\alpha$ is a $Z$-number if

(1) $$0 \leq r_n < \frac{1}{2} \quad \text{for all suffixes } n \geq 0.$$

Several years ago, a Japanese colleague proposed to me the problem whether such $Z$-numbers do in fact exist. I have not succeeded in solving this problem, but shall give here a number of incomplete results. In particular, it will be proved that the set of all $Z$-numbers is at most countable.

1

Assume that $\alpha$ is a $Z$-number. Evidently

$$g_{n+1} + r_{n+1} = \frac{3}{2}(g_n + r_n).$$

Here $g_n$ and $g_{n+1}$ are integers, while $r_n$ and $r_{n+1}$ lie in the interval

$$J = [0, \frac{1}{2}).$$

Hence one of the following two cases must hold.

(A) $g_n$ is an even number, hence $\frac{3}{2}g_n$ is an integer. Since

$$0 \leq \frac{3}{2}r_n < \frac{3}{4},$$

necessarily

$$g_{n+1} = \frac{3}{2}g_n \quad \text{and} \quad r_{n+1} = \frac{3}{2}r_n.$$
(B) \( g_n \) is an odd number and so both numbers \( \frac{3}{2}g_n \mp \frac{1}{2} \) are integers. Since \( \frac{3}{2}r_n + \frac{1}{2} \) cannot lie in \( J \), we now must have
\[
g_{n+1} = \frac{3}{2}g_n + \frac{1}{2} \quad \text{and} \quad r_{n+1} = \frac{3}{2}r_n - \frac{1}{2}.
\]
Put
\[
\epsilon_n = \begin{cases} 
0 & \text{if } g_n \text{ is even,} \\
1 & \text{if } g_n \text{ is odd.}
\end{cases}
\]
The two cases (A) and (B) can then be combined in the one formula
\[
g_{n+1} = \frac{3}{2}g_n + \frac{1}{2}\epsilon_n, \quad r_{n+1} = \frac{3}{2}r_n - \frac{1}{2}\epsilon_n.
\]
We also see that the case (A) can hold only if
\[
0 \leq r_n < \frac{1}{3}
\]
and case (B) if
\[
\frac{1}{3} \leq r_n < \frac{1}{2}.
\]
Hence \( \epsilon_n \) may also be defined by
\[
\epsilon_n = \begin{cases} 
0 & \text{if } 0 \leq r_n < \frac{1}{3}, \\
1 & \text{if } \frac{1}{3} \leq r_n < \frac{1}{2}.
\end{cases}
\]

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From \((2)\),
\[
g_0 = -\frac{1}{3}\epsilon_0 + \frac{2}{3}g_1, \quad g_1 = -\frac{1}{3}\epsilon_1 + \frac{2}{3}g_2, \cdots, \quad g_{n-1} = -\frac{1}{3}\epsilon_{n-1} + \frac{2}{3}g_n.
\]
Since
\[
g_0 + r_0 = \left(\frac{3}{2}\right)^n(g_n + r_n),
\]
it follows from these equations that
\[
g_0 = -\frac{1}{3}(\epsilon_0 + \frac{2}{3}\epsilon_1 + \left(\frac{2}{3}\right)^2\epsilon_2 + \cdots + \left(\frac{2}{3}\right)^{n-1}\epsilon_{n-1}) + \left(\frac{2}{3}\right)^n g_n
\]
and similarly also
\[
r_0 = +\frac{1}{3}(\epsilon_0 + \frac{2}{3}\epsilon_1 + \left(\frac{2}{3}\right)^2\epsilon_2 + \cdots + \left(\frac{2}{3}\right)^{n-1}\epsilon_{n-1}) + \left(\frac{2}{3}\right)^n r_n.
\]
These equations can be generalised. For this purpose put
\[
\alpha_0 = \alpha \quad \text{and} \quad \alpha_m = \left(\frac{3}{2}\right)^m \alpha.
\]
Then
\[
\left(\frac{3}{2}\right)^n(g_m + r_m) = \left(\frac{3}{2}\right)^n \alpha_m = \left(\frac{3}{2}\right)^{m+n} \alpha = g_{m+n} + r_{m+n},
\]
and it follows in analogy to \((3)\) and \((4)\) that for all suffixes \( m \) and \( n \),
\[
g_m = -\frac{1}{3}(\epsilon_m + \frac{2}{3}\epsilon_{m+1} + \left(\frac{2}{3}\right)^2\epsilon_{m+2} + \cdots + \left(\frac{2}{3}\right)^{n-1}\epsilon_{m+n-1}) + \left(\frac{2}{3}\right)^n g_{m+n}
\]
and
\[
r_m = +\frac{1}{3}(\epsilon_m + \frac{2}{3}\epsilon_{m+1} + \left(\frac{2}{3}\right)^2\epsilon_{m+2} + \cdots + \left(\frac{2}{3}\right)^{n-1}\epsilon_{m+n-1}) + \left(\frac{2}{3}\right)^n r_{m+n}.
\]
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The formula (6) for $r_m$ immediately implies a convergent series for this number. For all $r_{m+n}$ lie in the interval $J$, while the factor $(\frac{2}{3})^n$ tends to zero as $n$ tends to infinity. It follows therefore that for all suffixes $m \geq 0$,

$$3r_m = \varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + (\frac{2}{3})^2\varepsilon_{m+2} + \cdots$$

and in particular,

$$3r_0 = \varepsilon_0 + \frac{2}{3}\varepsilon_1 + (\frac{2}{3})^2\varepsilon_2 + \cdots$$

Here the convergence is in the sense of ordinary real analysis.

Consider next the formula (5) for $g_m$. The last term $(\frac{2}{3})^ng_{m+n}$ of this formula is a rational number the numerator of which is divisible by at least the $n$-th power of 2. In the so-called 2-adic analysis in the rational number field one considers numbers as small if they are divisible by a high power of 2 in the numerator, and as large if such a power of 2 occurs in the denominator. In this 2-adic sense the sequence of numbers $(\frac{2}{3})^ng_{m+n}$ tends to zero as $n$ tends to infinity. We may therefore write

$$-3g_m = \varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + (\frac{2}{3})^2\varepsilon_{m+2} + \cdots$$

in the 2-adic sense, and in particular,

$$-3g_0 = \varepsilon_0 + \frac{2}{3}\varepsilon_1 + (\frac{2}{3})^2\varepsilon_2 + \cdots$$

It is rather interesting that the same series converges in two different senses and to two different limits.

From this we can already deduce the fact the set of all Z-numbers is at most countable. For if the integer $g_0 \geq 0$ is given, then, by § 1, the corresponding sequence of integers $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ is determined uniquely, and so, by (8), also the fractional part $r_0$. We may express this result as follows.

(11) For any given non-negative integer $g_0$ there exists at most one Z-number in the interval $[g_0, g_0+1)$, and this Z-number lies in fact in the first half $[g_0, g_0+\frac{1}{2})$ of this interval.

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Much more can be said about the possible Z-numbers and their integral parts $g_0$.

All the fractional parts $r_m$, where $r = 0, 1, 2, \cdots$, lie by construction in the interval $J = [0, \frac{1}{2})$. This means by (7) that for every suffix $m$ the inequality

$$\varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + (\frac{2}{3})^2\varepsilon_{m+2} + \cdots < \frac{3}{2}$$
is satisfied. In this set of inequalities each of the numbers $\varepsilon_m, \varepsilon_{m+1}, \varepsilon_{m+2}, \cdots$ can assume only either of the two values 0 or 1.

It is then, firstly, immediately clear that for no $m$ simultaneously

$$\varepsilon_m = \varepsilon_{m+1} = 1.$$ 

For this would imply that

$$\varepsilon_m + 2\frac{2}{3}\varepsilon_{m+1} + \left(\frac{2}{3}\right)^2\varepsilon_{m+2} + \cdots \geq \frac{5}{3} > \frac{3}{2},$$

contrary to (12). Therefore

$$\text{(13)} \quad \text{if } m < n \text{ and } \varepsilon_m = \varepsilon_n = 1, \text{ then } n \geq m + 2.$$ 

From the inequalities (12) one can deduce restrictions on those suffixes $m$ for which simultaneously $\varepsilon_m = \varepsilon_{m+2} = 1, \varepsilon_{m+1} = 0$. We omit this discussion because no use will be made of the results so obtained.

5

Denote from now on by

$$M = \{m_1, m_2, m_3, \cdots\}, \quad \text{where} \quad 0 \leq m_1 < m_2 < m_3 < \cdots,$$

the set of all suffixes $m$ for which $\varepsilon_m = 1$. Thus

$$\varepsilon_m = \begin{cases} 1 & \text{if } m \in M, \\ 0 & \text{if } m \notin M. \end{cases}$$

In other words, $g_m$ is even or odd according as to whether $m$ is, or is not, an element of $M$.

Further put

$$G_k = g_{m_k} \quad (k = 1, 2, 3, \cdots),$$

so that all the $G_k$ are odd.

On applying the equation (5) with

$$m = m_k \quad \text{and} \quad m + n = m_{k+1},$$

thus with

$$\varepsilon_m = 1, \quad \varepsilon_{m+1} = \varepsilon_{m+2} = \cdots = \varepsilon_{m+n-1} = 0,$$

it follows that

$$G_k = -\frac{1}{3} + \left(\frac{2}{3}\right)^{m_{k+1}-m_k} G_{k+1},$$

hence that

$$G_{k+1} = \left(\frac{3}{2}\right)^{m_{k+1}-m_k-1} \frac{3G_k + 1}{2}. \quad \text{(14)}$$

This formula leads to the following algorithm connected with our problem.
We shall use the notation
\[ 2^a||H \]
to denote that \( H \) is divisible by \( 2^a \), but not by \( 2^{a+1} \).

\[ \text{6} \]

Put
\[ a_k = m_{k+1} - m_k - 1, \quad H_k = \frac{3G_k + 1}{2}. \]

Then, by (14), the following properties hold.

For every \( k \geq 1 \),

\[ \text{(16) } G_k \text{ is odd; } H_k \text{ is even; } a_k \geq 1; \quad 2^{a_k} || H_k; \text{ and } G_{k+1} = (\frac{3}{2})^{a_k} H_k \text{ is odd.} \]

Thus, starting with any odd integer \( G_1 \), these formulae allow to determine successively the integers

\[ H_1, a_1; \quad G_2, H_2, a_2; \quad G_3, H_3, a_3; \ldots \]

If \( G_1 \) was the integral part of a \( Z \)-number, then this algorithm can be continued indefinitely. It thus provides a necessary (but not a sufficient) condition for \( G_1 \) to be the integral part of a \( Z \)-number.

By way of example, if we start with \( G_1 = 13 \), we obtain the following sequence of integers.

\[
\begin{align*}
G_1 &= 13 & H_1 &= 20 & a_1 &= 2 \\
G_2 &= 45 & H_2 &= 68 & a_2 &= 2 \\
G_3 &= 153 & H_3 &= 230 & a_3 &= 1 \\
G_4 &= 345 & H_4 &= 518 & a_4 &= 1 \\
G_5 &= 777 & H_5 &= 1166 & a_5 &= 1 \\
G_6 &= 1749 & H_6 &= 2624 & a_6 &= 6 \\
G_7 &= 29889 & H_7 &= 44834 & a_7 &= 1 \\
G_8 &= 67251 & H_8 &= 100877.
\end{align*}
\]

Since \( H_8 \) is odd, the algorithm breaks off, and there is no \( Z \)-number between 13 and 14.

In spite of much computer work, no integer \( G_1 \) is known for which the algorithm does not break off. It is thus highly problematical whether there do in fact exist \( Z \)-numbers.

\[ \text{7} \]

If the existence of \( Z \)-numbers is assumed, further properties of such numbers can be obtained.
Let us deal with the possible frequency of $Z$-numbers! We have already seen that there can be at most one $Z$-number in each interval between consecutive integers $g$ and $g+1$ where $g \geq 0$. Thus, for $x > 0$, there are not more than $x+1$ $Z$-numbers between 0 and $x$. This estimate can now be replaced by a stronger one.

Let us first consider $Z$-numbers with odd integral parts, say with the integral part $G_1$. Put

$$b_k = a_k + 1 \quad \text{and} \quad c_k = a_k - 1 \quad (k = 1, 2, 3, \ldots),$$

so that by (16),

$$b_k \geq 2 \quad \text{and} \quad c_k \geq 0 \quad \text{for all} \quad k.$$

By (15) and (16),

$$G_k = -\frac{1}{3} + (\frac{2}{3})^{b_k}G_{k+1}.$$

On applying this equation repeatedly, we find that

$$G_1 = -\frac{1}{3} \left(1 + (\frac{2}{3})^{b_1} + (\frac{2}{3})^{b_1+b_2} + \cdots + (\frac{2}{3})^{b_1+b_2+\cdots+b_n} \right) + (\frac{2}{3})^{b_1+b_2+\cdots+b_{n+1}}G_{n+2}.$$

Here

$$B_n = -\frac{1}{3} \left(1 + (\frac{2}{3})^{b_1} + (\frac{2}{3})^{b_1+b_2} + \cdots + (\frac{2}{3})^{b_1+b_2+\cdots+b_n} \right)$$

is a rational number with an odd numerator and with a denominator which is a power of 3.

8

Let now $t$ be an arbitrarily large positive integer. For the given $Z$-number there exists just one suffix $n$ such that

$$b_1 + b_2 + \cdots + b_n \leq t < b_1 + b_2 + \cdots + b_{n+1}.$$

There further is a unique integer $s_n$ satisfying

$$1 \leq s_n \leq 2^t - 1$$

such that

$$B_n \equiv s_n \pmod{2^t},$$

i.e. that the numerator of $B_n - s_n$ is divisible by $2^t$. It is then clear from (17) that also

$$G_1 \equiv s_n \pmod{2^t}.$$

The rational number $B_n$, and so also the integer $s_n$, depend only on $t$ and on the ordered set of integers $b_1, b_2, \ldots, b_n$. Denote by $T(t)$ the number of ordered sets of integers $n, b_1, b_2, \ldots, b_n$ which satisfy the left-hand inequality (18). This number $T(t)$ is then also the number of all residue classes $s_n$ (mod $2^t$) in which there can lie odd integral parts $G_1$ of $Z$-numbers.
One can easily obtain an upper bound for \( T(t) \). The left-hand inequality (18) is equivalent to the inequality
\[
c_1 + c_2 + \cdots + c_n \leq t - 2n;
\]
hence \( T(t) \) may also be defined as the number of ordered solutions \( n, c_1, c_2, \cdots, c_n \) of this inequality where now \( c_1, c_2, \cdots, c_n \) may run independently over all non-negative integers. For each separate value of \( n \), this inequality has
\[
\binom{[t-2n]+n}{n} = \binom{t-n}{n}
\]
solutions, and hence, summing over \( n \),
\[
T(t) = \binom{t-1}{1} + \binom{t-2}{2} + \binom{t-3}{3} + \cdots
\]
where all terms after the \( \left\lfloor \frac{t}{2} \right\rfloor \)-th vanish.

This formula may be written as
\[
T(t) + 1 = \sum_{n=0}^{t} \binom{t-n}{n} = \sum_{n=0}^{t} \binom{n}{t-n}.
\]

By the binomial theorem, it implies that \( T(t) + 1 \) is the coefficient of \( z^t \) in the power series in powers of \( z \) for
\[
\sum_{n=0}^{t} \{z(1+z)\}^n = \frac{1 - \{z(1+z)\}^{t+1}}{1-z(1+z)},
\]
and hence \( T(t) + 1 \) is also the coefficient of \( z^t \) in the power series for
\[
f(z) = \frac{1}{1-z-z^2}.
\]

Put
\[
A = \frac{1+\sqrt{5}}{2}, \quad B = \frac{1-\sqrt{5}}{2},
\]
so that \( A + B = 1, \quad AB = -1, \quad A - B = \sqrt{5}. \)

Then
\[
1 - z - z^2 = (1 - Az)(1 - Bz) \quad \text{and} \quad f(z) = \frac{1}{\sqrt{5}} \left( \frac{A}{1-Az} - \frac{B}{1-Bz} \right).
\]

On developing here \( f(z) \) into a series in powers of \( z \), it follows at once that
\[
T(t) = \frac{1}{\sqrt{5}} \{A^{t+1} - B^{t+1}\} - 1.
\]

Actually, \( T(t) + 1 \) is the \( (t+1) \)-st term of the well known Fibonacci sequence.
Since trivially $B^{t+1}$ has the limit 0 as $t$ tends to infinity, and since further $A < \sqrt{5}$, it also follows from (20) that, for sufficiently large $t$,

\begin{equation}
T(t) \leq \left(\frac{1+\sqrt{5}}{2}\right)^t.
\end{equation}

9

By the definition of $T(t)$, there are $T(t)$ distinct residue classes (mod $2^t$) in which the integral part $G_1$ of a $Z$-number can lie when it is odd.

Consider next a $Z$-number $\alpha = g_0 + r_0$ with even integral part $g_0$, say

$$2^m || g_0.$$ 

Then

$$\alpha, \frac{3}{2}\alpha, (\frac{3}{2})^2\alpha, \ldots, (\frac{3}{2})^m\alpha$$

likewise are $Z$-numbers, and they have the integral parts

$$g_0, \frac{3}{2}g_0, (\frac{3}{2})^2g_0, \ldots, (\frac{3}{2})^m g_0,$$

respectively. Here $(\frac{3}{2})^m g_0 = G_1$ say, is an odd integer divisible by $3^m$, and

$$g_0 = (\frac{3}{2})^m G_1, \quad \frac{3}{2}g_0 = (\frac{3}{2})^{m-1} G_1, \ldots, \quad (\frac{3}{2})^m g_0 = G_1.$$ 

These $m+1$ products lie in the residue classes

\begin{equation}
(\frac{3}{2})^\mu G_1 \pmod{2^t},
\end{equation}

respectively, where $\mu$ runs over the successive values $\mu = m, m-1, m-2, \ldots, 1, 0$. If $\mu \geq t$, then $(\frac{3}{2})^\mu G_1$ lies in the residue class $\equiv 0 \pmod{2^t}$.

Thus to every odd residue class $G_1 \pmod{2^t}$ containing the integral part of a $Z$-number there correspond at most $t$ even residue classes (22) in which there are likewise integral parts of $Z$-numbers.

\begin{equation}
This implies that there cannot be more than
\end{equation}

$$(t+1)T(t)$$

odd or even residue classes (mod $2^t$) containing the integral part of a $Z$-number.

10

Trivially,

$$\frac{1+\sqrt{5}}{2} < 20.7.$$ 

Thus, as soon as $t$ is sufficiently large, it follows from (21) that there exist at most
odd or even residue classes \((\text{mod } 2^t)\) in which there is the integral part of at least one \(Z\)-number.

Denote now by \(x\) a sufficiently large positive integer, and choose the integer \(t\) such that
\[
2^t \leq x - 1 < 2^{t+1}.
\]
Then every residue class \((\text{mod } 2^t)\) contains at most two integers \(\leq x - 1\). Hence there can be at most \(\text{two} Z\)-numbers not greater than \(x\) the integral parts of which lie in this residue class. By (23), the number of residue classes which need be considered is only
\[
20.7 \cdot t^{-1} < \frac{1}{2} x^{0.7}.
\]
We obtain therefore the following result.

(24) For sufficiently large \(x\) there are at most
\[
x^{0.7}
\]
\(Z\)-numbers satisfying
\[
0 \leq \alpha \leq x.
\]

This paper dealt with the numbers \(\alpha\) for which the fractional parts \(r_n\) defined in § 1 satisfied the inequalities
\[
0 \leq r_n < \frac{1}{2} \quad (n = 0, 1, 2, \cdots).
\]
It is possible to establish a similar theory if all the \(r_n\) are assumed to lie in some other subinterval \([c, c+\frac{1}{2})\) of \([0, 1)\). It would be very interesting if a similar theory could be established for subintervals of smaller length, or perhaps even of arbitrarily small length.

Naturally, one can consider analogous problems for the products
\[
\alpha \left(\frac{p}{q}\right)^n \quad (n = 0, 1, 2, \cdots)
\]
where \(\alpha\) is again a positive number, and \(p\) and \(q\) are integers satisfying
\[
p > q \geq 2, \quad (p, q) = 1.
\]

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