Matematica. — *On formal power series as integrals of algebraic
differential equations.* Nota di Kurt Mahler, presentata (*) dal Socio
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In memory of my dear friend Jan Popken.

Riassunto. Si stabilisce l'esistenza di due costanti reali positive \( \gamma_1, \gamma_2 \) siffatte che, per una qualsiasi serie formale di potenze \( \sum_{h=0}^{\infty} f_h z^h \) a coefficienti \( f_h \) complessi che sia soluzione di una qualche equazione differenziale algebraica, debba risultare \( |f_h| \sim \gamma_1 (h!)^{\gamma_2} \) per \( h = 0, 1, 2, \ldots \).

The following result will be proved. Let

\[
f = \sum_{h=0}^{\infty} f_h z^h
\]

be a formal power series with complex coefficients which satisfies any algebraic differential equation. Then two positive constants \( \gamma_1 \) and \( \gamma_2 \) exist such that

\[
f_h \sim \gamma_1 (h!)^{\gamma_2}
\]

for all \( h \). This estimate is the best possible. For if \( n \) is any positive integer, the series

\[
\sum_{h=0}^{\infty} (h!)^n z^h
\]

is known to satisfy a linear differential equation with coefficients that are polynomials in \( z \).

1. Denote by \( K \) an arbitrary subfield of the complex number field \( \mathbb{C} \), and by \( K^* \) the ring of all formal power series

\[
f = \sum_{h=0}^{\infty} f_h z^h \quad , \quad g = \sum_{h=0}^{\infty} g_h z^h \quad , \quad \text{etc.,}
\]

with coefficients \( f_h, g_h, \ldots \) in \( K \). Here sum and product are as usual defined by

\[
f + g = \sum_{h=0}^{\infty} (f_h + g_h) z^h \quad , \quad fg = \sum_{h=0}^{\infty} \left( \sum_{k=0}^{h} f_k g_{h-k} \right) z^h,
\]

and the elements \( a \) of \( K \) are identified with the special series

\[
a = a \cdot \sum_{h=0}^{\infty} z^h
\]

and play the role of constants.

(*) Nella seduta del 20 febbraio 1971.
Differentiation in $K^*$ is defined formally by
\[ \frac{d^k f}{dz^k} = f^{(k)} = \sum_{h=k}^{\infty} h(h-1) \cdots (h-k+1)f_h z^{h-k}, \]
a notation used also for $k = 0$ when
\[ f^{(0)} = f. \]
In particular,
\[ \frac{df}{dz} = 0 \quad \text{if and only if} \quad f = a \in K. \]
The usual rules for the derivatives of sum, difference, and product hold also in $K^*$.

An important mapping from $K^*$ into $K$ is defined by the formal substitution $z = 0$. For this substitution we use the notation
\[ f(0) = f|_{z=0} = f_0. \]
More generally
\[ f^{(k)}(0) = f^{(k)}|_{z=0} = k! f_k. \]

2. This paper is concerned with power series
\[ f = \sum_{h=0}^{\infty} f_h z^h \]
in $K^*$ which satisfy any algebraic differential equation
\[ (F) \quad F(\langle \omega \rangle) = F(\langle \omega, \omega', \omega'', \cdots, \omega^{(m)} \rangle) = 0. \]
Here $F(\langle \omega, \omega_0, \omega_1, \cdots, \omega_m \rangle) \equiv 0$ denotes a polynomial in the indeterminates $\omega, \omega_0, \omega_1, \cdots, \omega_m$ with coefficients in some extension field of $K$. By a well-known method from linear algebra $f$ can then be shown to satisfy also an algebraic differential equation $(F)$ with coefficients in $K$; only this case will therefore from now on be considered.

Put
\[ F_{\mu}(\langle \omega, \omega_0, \omega_1, \cdots, \omega_m \rangle) = \frac{\partial}{\partial \omega^{(\mu)}} F(\langle \omega, \omega_0, \omega_1, \cdots, \omega_m \rangle) \quad (\mu = 0, 1, \cdots, m), \]
and
\[ F_{(\mu)}(\langle \omega \rangle) = F_{\mu}(\langle \omega, \omega', \omega'', \cdots, \omega^{(m)} \rangle) \quad (\mu = 0, 1, \cdots, m), \]
where $\omega$ denotes an indeterminate element of $K^*$. There is no loss of generality in assuming that both
\[ F_{(m)}(\langle \omega \rangle) \equiv 0 \tag{1} \]
and
\[ F_{(m)}(\langle f \rangle) \equiv 0 \tag{2} \]
The integer \( m \geq 0 \) is thus the exact order of the differential equation (F); when \( m = 0 \), (F) becomes an algebraic equation for \( f \), a case which need not be excluded.

3. The differential operator \( F((w)) \) has the explicit form

\[
F((w)) = \sum_{(v)} p(v)(z) w^{(v_1)} \cdots w^{(v_n)}.
\]

Here the summation

\[
\sum_{(v)}
\]

extends over all ordered systems

\[
(v) = (v_1, \cdots, v_n)
\]

of integers for which

\[
0 \leq v_1 \leq m, \cdots, 0 \leq v_n \leq m; \quad v_1 \leq v_2 \leq \cdots \leq v_n; \quad 0 \leq N \leq n,
\]

where \( n \) is a fixed positive integer, and the \( p(v)(z) \) are polynomials in \( K[z] \). The integer \( N \) may vary with the system \((v)\), and there is just one improper system \((v)\) denoted by \((\omega)\) for which \( N = 0 \). The term in (3) corresponding to \((v) = (\omega)\) has the form

\[
p(\omega)(z)
\]

and thus has no factors \( w^{(v)} \), but is a polynomial in \( z \) alone.

4. An explicit expression for the successive derivatives

\[
F^{(h)}((w)) = \left( \frac{d}{dz} \right)^h F((w)) \quad (h = 1, 2, 3, \cdots)
\]

of \( F((w)) \) can be obtained by means of the following simple lemma.

**Lemma 1:** Let \( h \geq 1 \) and \( N \geq 0 \) be arbitrary integers, and let

\[
w_0, w_1, \cdots, w_N
\]

be any \( N + 1 \) elements of \( K^* \). Then

\[
\left( \frac{d}{dz} \right)^h (w_0 w_1 \cdots w_N) = h! \sum \frac{w_0^{(\lambda_0)}}{\lambda_0!} \frac{w_1^{(\lambda_1)}}{\lambda_1!} \cdots \frac{w_N^{(\lambda_N)}}{\lambda_N!},
\]

where the summation extends over all ordered systems of \( N + 1 \) integers \( \lambda_0, \lambda_1, \cdots, \lambda_N \) satisfying

\[
\lambda_0 \geq 0, \lambda_1 \geq 0, \cdots, \lambda_N \geq 0; \quad \lambda_0 + \lambda_1 + \cdots + \lambda_N = h.
\]
Proof. The assertion is evident when $h = 1$. Assume it has already been established for some $h \geq 1$. We now show that then it holds also for $h + 1$ and hence is always true.

On differentiating (5),
\[
\left( \frac{d}{dz} \right)^{h+1} (w_0 w_1 \cdots w_N) = h! \sum_{\lambda_0, \lambda_1, \cdots, \lambda_N} \frac{\lambda_0!}{\lambda_0!} \frac{\lambda_1!}{\lambda_1!} \cdots \frac{\lambda_N!}{\lambda_N!} w_0^{(\lambda_0)} w_1^{(\lambda_1)} \cdots w_N^{(\lambda_N)} =
\]
\[
= h! \sum_{\mu_0, \mu_1, \cdots, \mu_N} \left( \sum_{v=0}^{N} \frac{\mu_0!}{\mu_0!} \frac{\mu_1!}{\mu_1!} \cdots \frac{\mu_N!}{\mu_N!} \right) w_0^{(\mu_0)} w_1^{(\mu_1)} \cdots w_N^{(\mu_N)},
\]
where the new summation extends over all ordered systems of $N + 1$ integers $\mu_0, \mu_1, \cdots, \mu_N$ satisfying
\[
\mu_0 \geq 0, \mu_1 \geq 0, \cdots, \mu_N \geq 0 ; \mu_1 + \mu_0 + \cdots + \mu_N = h + 1.
\]
Since thus,
\[
h! \sum_{v=1}^{N} \mu_v = (h + 1)!,
\]
the assertion follows.

5. Apply Lemma 1 to all the separate terms
\[
P^{(n)}(z) w^{(\mu_1)} w^{(\mu_2)}
\]
in the formula (3) for $F((w))$. It follows then that
\[
F^{(h)}((w)) = h! \sum_{(\sigma)} \sum_{[\lambda]} P^{(h)}_{(\sigma)}(z) \frac{\lambda_0!}{\lambda_0!} \frac{\lambda_1!}{\lambda_1!} \cdots \frac{\lambda_N!}{\lambda_N!} w_0^{(\mu_0 + \lambda_0)} w_1^{(\mu_1 + \lambda_1)} \cdots w_N^{(\mu_N + \lambda_N)}.
\]
Here the inner summation
\[
\sum_{[\lambda]}
\]
extends over all ordered systems $[\lambda] = [\lambda_0, \lambda_1, \cdots, \lambda_N]$ of $N + 1$ integers for which
\[
\lambda_0 \geq 0, \lambda_1 \geq 0, \cdots, \lambda_N \geq 0 ; \lambda_0 + \lambda_1 + \cdots + \lambda_N = h,
\]
and $N$ denotes the same integer as for the system $(x)$. There is exactly one term
\[
P_{(\mu)}^h(z)
\]
in the development (6) for which $N = 0$. This term does not involve $w$, and it vanishes as soon as $h$ exceeds the degree of the polynomial $P_{(\mu)}^h(z)$.

6. From its definition, $F^{(h)}((w))$ is a polynomial in $z$ and $w, w', \cdots, w^{(h+m)}$. We next show that, for sufficiently large $k$, $F^{(h)}((w))$ is linear in the derivative $w^{(k)}$.

Let $j$ be any integer in the interval
\[
o \leq j \leq \left\lfloor \frac{h-1}{2} \right\rfloor,
\]
and define \( k \) in terms of \( h \) by
\[
k = h + m - j.
\]
Further denote by
\[
F^{(h,k)}(\vec{\omega}) \cdot \omega^{(k)}
\]
the sum of all terms on the right-hand side of (6) which have at least one factor \( \omega^{(k)} \), and denote by
\[
F^{(h,k)}_{(\omega)}(\vec{\omega}) \cdot \omega^{(k)}
\]
the sum of all those contributions to \( F^{(h,k)}(\vec{\omega}) \omega^{(k)} \) which are obtained from the \( h \)-th derivative
\[
\left( \frac{d}{dz} \right)^h (p^{(x)}(z) \omega^{(x_1)} \cdots \omega^{(x_s)}) = h! \sum_{|k|} \frac{p^{(x)}_{(\omega)}(z)}{k_0!} \frac{\omega^{(x_1)}(z)}{k_1!} \cdots \frac{\omega^{(x_s)}(z)}{k_s!},
\]
of the term
\[
p^{(x)}(z) \omega^{(x_1)} \cdots \omega^{(x_s)},
\]
= \( t_{(\omega)} \) say,
in the representation (3) of \( F(\vec{\omega}) \). It is then clear that
\[
F^{(h,k)}(\vec{\omega}) = \sum_{(\omega)} F^{(h,k)}_{(\omega)}(\vec{\omega}),
\]
and that further
\[
F^{(h,k)}_{(\omega_0)}(\vec{\omega}) = 0.
\]
Hence there are non-zero contributions only from those terms \( t_{(\omega)} \) for which
\[
|x| = n \quad \text{and therefore} \quad 1 \leq N \leq n.
\]

7. Let now \( \nu \) be any element of the set \( \{1, 2, \ldots, N\} \), and let \( \nu' \) be any element of this set which is distinct from \( \nu \). It is obvious that the binomial coefficient
\[
\binom{h}{k - \nu}\nu
\]
vanishes if either
\[
k - \nu < 0 \quad \text{or} \quad k - \nu > h.
\]
It suffices therefore to consider those suffixes \( \nu \) for which
\[
o \leq k - \nu \leq h.
\]
Such suffixes will be said to be \textit{admissible}.

To every admissible suffix \( \nu \) there exist systems \( [\lambda] = [\lambda_0, \lambda_1, \ldots, \lambda_N] \) of \( N + 1 \) integers satisfying both
\[
\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_N \geq 0 \quad \text{and} \quad \lambda_0 + \lambda_1 + \cdots + \lambda_N = h
\]
and
\begin{equation}
\tag{10}
z_v + \lambda_v = k.
\end{equation}
Hence, by the definitions of \(j\) and \(k\),
\[\lambda_v = k - z_v = (h - j) + (m - z_v) \geq h - j > \frac{h}{2},\]
and therefore
\[\lambda_v < \frac{h}{2}, \quad z_v + \lambda_v < \frac{h}{2} + m = h + m - \frac{h}{2} \leq h + m - j = k.\]
It follows that the corresponding term
\[h! \frac{\rho^{(k_d)}(z)}{\lambda_d!} \frac{\xi^{(\lambda_1 + \lambda_2)}}{\lambda_1!} \cdots \frac{\xi^{(\lambda_s + \lambda_s)}}{\lambda_s!},\]
\[= T_{(\omega),[k]}\]say,
on the right-hand side of (8) has one and only one factor \(\xi^{(\lambda_i)}\). Hence the contribution from \(T_{(\omega),[k]}\) to \(F^{(k_0)}(\omega)\) is equal to
\[\frac{\sum_{[k]} T_{(\omega),[k]}}{z^{(\lambda_i)}} = \frac{h!}{\lambda_v!} \frac{\rho^{(k_d)}(z)}{\lambda_d!} \prod_{v} \frac{\xi^{(\lambda_v + \lambda_v)}}{\lambda_v!}.\]
On the other hand, by Lemma 1, also
\[\left( \frac{d}{dz} \right)^{h-k+z_v} \left( \rho^{(k_d)}(z) \prod_v \xi^{(\lambda_v)} \right) = \left( h - k + z_v \right) \sum_{[k]} \left( \frac{\rho^{(k_d)}(z)}{\lambda_d!} \prod_v \frac{\xi^{(\lambda_v + \lambda_v)}}{\lambda_v!} \right)\]
where the summation \(\sum_{[k]}\) is extended only over those systems \([\lambda]\) which have both properties (7) and (10). Therefore
\[\sum_{[k]} \frac{\sum_{[k]} T_{(\omega),[k]}}{z^{(\lambda_i)}} = \left( \frac{d}{dz} \right)^{h-k+z_v} \left( \rho^{(k_d)}(z) \prod_v \xi^{(\lambda_v)} \right) = \left( \frac{d}{dz} \right)^{h-k+z_v} \left( \rho^{(k_d)}(z) \prod_v \xi^{(\lambda_v)} \right),\]
whence, on summing over \(v = 1, 2, \cdots, N\),
\[F^{(k_0)}(\omega) = \sum_{v=1}^{N} \left( \frac{d}{dz} \right)^{h-k+z_v} \left( \rho^{(k_d)}(z) \prod_v \xi^{(\lambda_v)} \right).\]
Here the terms belonging to non-admissible suffixes \(v\) vanish on account of the factor \(\left( \frac{h}{h-z_v} \right) = 0.\)
The formula so obtained may also be written as
\[F^{(k_0)}(\omega) = \sum_{\mu=0}^{m} \left( \frac{d}{dz} \right)^{h-k+\mu} \left( \rho^{(k_d)}(z) \prod_v \xi^{(\lambda_v)} \right).\]
because
\[
\frac{\partial}{\partial w^{(\mu)}} \left( p^{(\nu)}(z) w^{(\nu_1)} \cdots w^{(\nu_N)} \right) = \sum_{\nu} \frac{\partial}{\partial w^{(\nu)}} \left( p^{(\nu)}(z) w^{(\nu_1)} \cdots w^{(\nu_N)} \right)
\]
where \( \nu \) in \( \sum \) runs over all suffixes \( 1, 2, \cdots, N \) which satisfy \( \nu = \mu \).

Finally, by (3) and (9),
\[
F^{(h,k)}(\langle w \rangle) = \sum_{\mu=0}^{m} \left( \frac{h}{k-\mu} \right) \left( \frac{d}{dz} \right)^{h-k+\mu} F^{(\mu)}(\langle w \rangle)
\]
where, as in § 2, \( F^{(\mu)}(\langle w \rangle) \) denotes the partial derivative of \( F(\langle w \rangle) \) with respect to \( w^{(\mu)} \). This formula is due to A. Hurwitz (1889) and S. Kakeya (1915).

8. The basic identities (6) and (11) hold for all elements \( w \) of \( K^* \).

We apply them now to the integral \( f \) of \( (F) \). We so firstly obtain the equations
\[
F^{(h)}(\langle f \rangle) = h! \sum_{(\nu)} \sum_{\lambda} \left( \frac{h}{\lambda} \right) \left( \frac{d}{dz} \right)^{h-k+\mu} F^{(\mu)}(\langle f \rangle)
\]
and secondly, for all \( h = 1, 2, 3, \cdots \) and all \( j = 0, 1, \cdots, \frac{h-1}{2} \), find that
\[
F^{(h,k)}(\langle f \rangle) = \sum_{\mu=0}^{m} \left( \frac{h}{k-\mu} \right) \left( \frac{d}{dz} \right)^{h-k+\mu} F^{(\mu)}(\langle f \rangle),
\]
a formula which gives the coefficients of \( f^{(h)} = f^{(h+m-j)} \) in (12).

In (12) and (13) we finally put \( z = 0 \). Since
\[
\frac{p^{(\nu)}(z)}{\nu_1!} \bigg|_{z=0} = \frac{p^{(\nu)}(0)}{\nu_1!} \quad \text{and} \quad f^{(h)} \bigg|_{z=0} = h! f_h,
\]
this leads to the equations
\[
\frac{h!}{\sum_{(\nu)} \sum_{\lambda} \left( \frac{h}{\lambda} \right) \left( \frac{p^{(\nu)}(0)}{\nu_1!} \right) \left( \frac{x_1+\lambda_1}{\lambda_1!} \right) \cdots \left( \frac{x_n+\lambda_n}{\lambda_n!} \right) \cdot f^{(h,k)} = 0,}
\]
(\( h = 1, 2, 3, \cdots \)).

Furthermore, the coefficient of \( h! f_h \) on the left hand side is given by
\[
F^{(h,k)}(\langle f \rangle) \bigg|_{z=0} = \sum_{\mu=0}^{m} \left( \frac{h}{k-\mu} \right) \left( \frac{d}{dz} \right)^{h-k+\mu} F^{(\mu)}(\langle f \rangle) \bigg|_{z=0}.
\]
Here the expressions \( F^{(\mu)}(\langle f \rangle) \) are elements of \( K^* \), hence have the explicit form
\[
F^{(\mu)}(\langle f \rangle) = \sum_{h=0}^{\infty} F_{\mu,h} z^h
\]
with certain coefficients \( F_{\mu,h} \) in \( K \). Thus
\[
\left( \frac{d}{dz} \right)^{h-k+\mu} F^{(\mu)}(\langle f \rangle) \bigg|_{z=0} = (h-k+\mu) F_{\mu,k-h+\mu}
\]
for all \( h \) and \( \mu \).
Therefore further
\[
F^{(\ell,k)}(f) |_{\ell=0} = \sum_{\mu=0}^{m} \left( \begin{array}{c} h \\ k - \mu \end{array} \right) (h - \mu)! F_{k-\mu, \ell-\mu},
\]
whence, on replacing \( \mu \) by \( m - \mu \) and remembering that \( k - h + m - j \),
\[
(15) \quad F^{(\ell,k)}(f) |_{\ell=0} = \sum_{\mu=0}^{m} \left( \begin{array}{c} h \\ j - \mu \end{array} \right) (j - \mu)! F_{m-j, \ell-\mu}.
\]

All these expressions are polynomials in \( h \) with coefficients in \( K \). We can easily prove that they do not all vanish identically. For by hypothesis,
\[
(2) \quad F_{m}(f) = 0.
\]
Hence the coefficients \( F_{m,h} \) do not all vanish, and so there exists an integer
\[
t \geq 0,
\]
such that
\[
F_{m,0} = F_{m,1} = \cdots = F_{m,t-1} = 0, \quad \text{but} \quad F_{m,t} = 0.
\]
Thus, on choosing \( j = t \) and \( k = h + m - t \),
\[
F^{(\ell,k)}(f) |_{\ell=0} = \left( \begin{array}{c} h \\ t \end{array} \right) t! F_{m,t} + \sum_{\mu=0}^{m} \left( \begin{array}{c} h \\ t - \mu \end{array} \right) (t - \mu)! F_{m-j, \ell-\mu}
\]
is a polynomial in \( h \) of the exact degree \( t \) and certainly does not vanish identically.

It follows that there exists a smallest integer \( s \) satisfying
\[
o = s < t
\]
such that the polynomial \( (15) \) vanishes identically in \( h \) for \( j = 0, 1, \cdots, s - 1 \), but that the polynomial
\[
F^{(\ell,k)}(f) |_{\ell=0} = \sum_{\mu=0}^{m} \left( \begin{array}{c} h \\ s - \mu \end{array} \right) (s - \mu)! F_{m-j, s-\mu}, \quad \text{where} \quad k - h + m - s,
\]
is not identically zero. On changing over from \( h \) to \( k \), put
\[
(16) \quad a(k) = F^{(k-m+s,\ell)}(f) |_{\ell=0} = \sum_{\mu=0}^{m} \left( \begin{array}{c} k \\ m-s \end{array} \right) (s - \mu)! F_{m-j, s-\mu}.
\]
Then \( a(k) \) is now a polynomial in \( k \) which likewise does not vanish identically.

With \( s, k, \) and \( a(k) \) as just defined, we can now assert that for
\[
h = k - m + s \geq 2s + 1
\]
and hence for
\[
k \geq m + s + 1,
\]
the left-hand side of the equation (14) involves at most the coefficients

\[ f_0, f_1, \ldots, f_k \]

of \( f \), but is free of

\[ f_{k+1}, f_{k+2}, \ldots, f_{k+s} = f_{h^k} \cdot m. \]

Furthermore, on this left-hand side, \( k! f_k \) has the exact factor \( a(k) \).

9. The result just proved will enable us now to find both recursive equations and inequalities for the coefficients \( f_k \) of \( f \).

Put, firstly,

\[ a(k) = \begin{cases} k! & \text{if } k \geq h, \\ h^k & \text{if } k \leq h, \end{cases} \]

and, secondly,

\[ A(k) - a(k) a(k) \]

so that evidently all three expressions \( a(k) \), \( \beta(k) \), and \( A(k) \) are polynomials in \( k \) which do not vanish identically.

Thirdly, denote by

\[ \mathcal{\Phi}_k = \mathcal{\Phi}_k (f_0, f_1, \ldots, f_{k-1}) \]

the double sum

\[ \mathcal{\Phi}_k = \beta(k) \sum_{(\alpha)} \sum_{[\xi]^*} \beta_{(\alpha)}^{(\xi)} \frac{\lambda_1! \cdots \lambda_s!}{\lambda_1! \cdots \lambda_s!} f_{\lambda_1 + \lambda_2} \cdots f_{\lambda_s + \lambda_s}, \]

where the asterisk at \( \sum_{(\alpha)} \sum_{[\xi]^*} \) signifies that all terms having one of the factors

\[ f_{k+1}, f_{k+2}, \ldots, f_{k+s} \]

are to be omitted.

With this notation, we arrive at the basic recursive formula

\[ A(k) f_k = \mathcal{\Phi}_k (f_0, f_1, \ldots, f_{k-1}). \]

Here the polynomial \( A(k) \) is not identically zero, and hence, if \( k_0 \) denotes any sufficiently large integer, then

\[ A(k) \neq 0 \quad \text{if} \quad k \geq k_0. \]

Thus, whenever \( k \geq k_0 \), then the recursive formula (17) expresses \( f_k \) as a polynomial in \( f_0, f_1, \ldots, f_{k-1} \). By means of this representation, we shall now establish an upper estimate for \( |f_k| \).
10. For the moment, denote by \( z_0, z_1, \ldots, z_N \) arbitrary non-negative integers, and put

\[
\varphi_v = z_v ! \left( 1 - z \right)^{-(z_v + 1)} \quad (v = 0, 1, \ldots, N)
\]

so that

\[
\varphi_{v}^{(\lambda_v)} = (z_v + \lambda_v) ! \left( 1 - z \right)^{-(z_v + \lambda_v + 1)} \quad (v = 0, 1, \ldots, N)
\]

and also

\[
\frac{1}{h !} \left( \frac{d}{dz} \right)^h (\varphi_0 \varphi_1 \cdots \varphi_N) =
\]

\[
= z_0 ! z_1 ! \cdots z_N ! \left( \frac{z_0 + z_1 + \cdots + z_N + h + N}{z_0 + z_1 + \cdots + z_N + N} \right) \left( 1 - z \right)^{-(z_0 + z_1 + \cdots + z_N + h + N + 1)}.
\]

On the other hand, by Lemma 1,

\[
\frac{1}{h !} \left( \frac{d}{dz} \right)^h (\varphi_0 \varphi_1 \cdots \varphi_N) =
\]

\[
= \sum_{\lambda_0, \lambda_1, \ldots, \lambda_N} \frac{(z_0 + \lambda_0) !}{\lambda_0 !} \cdots \frac{(z_N + \lambda_N) !}{\lambda_N !} \left( 1 - z \right)^{-(z_0 + z_1 + \cdots + z_N + h + N + 1)}
\]

where the summation again extends over all systems \([\lambda]\) with the properties (7). On comparing these two formulae, we obtain the identity

\[
\sum_{[\lambda]} \frac{(z_0 + \lambda_0) !}{\lambda_0 !} \frac{(z_1 + \lambda_1) !}{\lambda_1 !} \cdots \frac{(z_N + \lambda_N) !}{\lambda_N !} = \left( \frac{z_0 + z_1 + \cdots + z_N + h + N}{z_0 + z_1 + \cdots + z_N + N} \right) z_0 ! z_1 ! \cdots z_N !.
\]

Here assume that

\[ z_0 = 0 ; \quad 0 \leq z_1 \leq m, \ldots, 0 \leq z_N \leq m ; \quad 1 \leq N \leq n. \]

Then

\[ 0 \leq z_0 + z_1 + \cdots + z_N + N \leq (m + 1) n, \]

and so the binomial coefficient

\[
\binom{z_0 + z_1 + \cdots + z_N + h + N}{z_0 + z_1 + \cdots + z_N + N} \leq (h + (m + 1) n)^{(m + 1) n}.
\]

The identity (19) implies therefore for all systems \((\lambda)\) as before that

\[
\sum_{[\lambda]} \frac{(z_1 + \lambda_1) !}{\lambda_1 !} \cdots \frac{(z_N + \lambda_N) !}{\lambda_N !} \leq m^{m_{\lambda}} \left( h + (m + 1) n \right)^{(m + 1) n}.
\]

11. The operator \( F(\varphi) \) depends on only finitely many polynomials \( p_v (z) \), and these together have only finitely many coefficients

\[
\frac{p_v^{(\lambda_0)} (0)}{\lambda_0 !}.
\]
The maximum

$$c_0 = \max_{(\nu), [\lambda]} \left| \frac{b^{(\ell_0)}(\nu)}{\lambda_0!} \right|$$

of the absolute values of all these coefficients is then a finite positive constant which, naturally, does not depend on \( k \).

On the right-hand side of the formula (17) for \( \varphi_k \), the double sum \( \sum_{(\nu)} \sum_{[\lambda]}^* \) is a subsum of the double sum \( \sum_{(\nu)} \sum_{[\lambda]} \). It follows then from (17) that

$$|A(k)||f_k| \leq |\beta(k)| \cdot c_0 \cdot m^{mn} \{ k + (m + 1) n - m + s \}^{(m + 1)n} \cdot \max_{(\nu), [\lambda]}^* \left| f_{\nu_1 + \lambda_1} \cdots f_{\nu_N + \lambda_N} \right|,$$

where \( \max^* \) is extended over all pairs of systems \((\nu), [\lambda]\) for which

$$1 \leq N \leq n ; \quad 0 \leq \nu_1 + \lambda_1 \leq k - 1, \cdots, 0 \leq \nu_N + \lambda_N \leq k - 1.$$

The estimate (22) can be slightly simplified. Let \( k_0 \) be the same constant as in (19). There exist then two further positive constants \( c_1 \) and \( c_2 \), both independent of \( k \), such that

$$|\beta(k)| \cdot \frac{c_0 \cdot m^{mn} \{ k + (m + 1) n - m + s \}^{(m + 1)n}}{A(k)} \leq k^{c_1} \quad \text{for} \quad k \geq k_0,$$

and hence also

$$|\beta(k)| \cdot \frac{c_0 \cdot m^{mn} \{ k + (m + 1) n - m + s \}^{(m + 1)n}}{A(k)} \leq k^{c_2} \quad \text{for} \quad k \geq k_0.$$

Next, with any two systems \((\nu)\) and \([\lambda]\) we can associate a further ordered system of \( N \) integers \( \{\nu\} = \{\nu_1, \cdots, \nu_N\} \) by putting

$$\nu_1 = \nu_1 + \lambda_1, \cdots, \nu_N = \nu_N + \lambda_N.$$

Then, by (23),

$$1 \leq N \leq n ; \quad 0 \leq \nu_1 \leq k - 1, \cdots, 0 \leq \nu_1 \leq k - 1.$$

Further, by the properties of \((\nu)\) and \([\lambda]\),

$$\nu_1 + \cdots + \nu_N = (\nu_1 + \cdots + \nu_N) + \lambda = k + (\nu_1 + \cdots + \nu_N - m + s),$$

and hence there exists a further positive constant \( c_3 \) independent of \( k \) such that

$$\nu_1 + \cdots + \nu_N \leq k + c_3.$$

It follows therefore finally from (22) and (24) that

$$|f_k| \leq k^{c_2} \cdot \max_{\{\nu\}} \left| f_{\nu_1} \cdots f_{\nu_N} \right| \quad \text{for} \quad k \geq k_0,$$

where the maximum is extended over all systems \( \{\nu\} \) with the properties (26) and (27).
12. We may now, without loss of generality, assume that

\[ k_0 > c_3 + 1. \]

Choose any \( k_0 \) positive numbers \( u_0, u_1, \ldots, u_{k_0-1} \) such that

\[ 0 < u_0 < u_1 < \cdots < u_{k_0-1}, \quad \text{and} \quad |f_k| \leq e^{u_k} \quad \text{for} \quad k = 0, 1, \ldots, k_0-1, \]

and then, for each suffix \( k \geq k_0 \), define recursively a number \( u_k \) by the equation

\[ u_k = c_2 \log k + \max_{\{v\} \in S_k} (u_{v_1} + \cdots + u_{v_N}). \]

Here \( \{v\} \) is to run again over all systems of integers with the properties (26) and (27). For use below, denote by \( S_k \) the set of all such systems \( \{v\} \).

We assert that with this definition of \( u_k \),

\[ |f_k| < e^{u_k} \quad \text{for all suffixes} \quad k \geq 0. \]

For this is certainly true for \( k \leq k_0 - 1 \), and it is for larger \( k \) a consequence of (28) and (31) because

\[ |f_k| \leq \exp \left( c_2 \log k + \max_{\{v\} \in S_{k+1}} (u_{v_1} + \cdots + u_{v_N}) \right) = e^{u_k}. \]

Let now again \( k \geq k_0 \), hence, by (29),

\[ k > c_3 + 1. \]

The recursive formula (31) implies then that

\[ u_{k+1} - u_k = c_2 \log \frac{k+1}{k} + \max_{\{v\} \in S_{k+1}} (u_{v_1} + \cdots + u_{v_N}) - \max_{\{v\} \in S_k} (u_{v_1} + \cdots + u_{v_N}). \]

Here \( S_k \) evidently is a subset of \( S_{k+1} \); the maximum over \( S_{k+1} \) is therefore not less than that over \( S_k \), and so (33) implies that

\[ u_{k+1} - u_k \geq c_2 \log \frac{k+1}{k} > 0 \quad \text{for} \quad k \geq k_0. \]

Together with the first inequalities (30), this proves that the numbers \( u_k \) form a strictly increasing sequence of positive numbers.

13. Consider now any system \( \{\pi\} = (\pi_1, \ldots, \pi_{N^*}) \) in \( S_{k+1} \) at which the maximum

\[ \max_{\{v\} \in S_{k+1}} (u_{v_1} + \cdots + u_{v_N^*}) = u_{\pi_1} + \cdots + u_{\pi_{N^*}}, \]

is attained. Since the numbers \( u_k \) are positive and strictly increasing, the suffixes \( \pi_1, \ldots, \pi_{N^*} \) cannot all be zero; moreover, since

\[ \pi_1 + \cdots + \pi_{N^*} \leq \tilde{k} + c_3 + 1 \quad \text{and} \quad \tilde{k} > c_3 + 1, \]
at most one of these suffixes can be as large as $k$. Denote by

$$\pi_{N^*} > 0$$

the largest of the suffixes $\pi_1, \ldots, \pi_{N^*}$, or one of them if several of these suffixes have the same maximum value. The other suffixes

$$\pi_1, \ldots, \pi_{N^*-1}$$

are then non-negative and less than $k$. Hence the system $\{v^0\} = \{v^0_1, \ldots, v^0_{N^*}\}$ defined by

$$N^0 = N^*, \quad v^0_1 = \pi_1, \ldots, v^0_{N^*-1} = \pi_{N^*-1}, \quad v^0_{N^*} = \pi_{N^*} - 1 \geq 0$$

belongs to the set $S_k$, and therefore

$$\max_{\{v\} \in S_k} (u_{v_1} + \cdots + u_{v_{N^*}}) > u_{\pi_1} + \cdots + u_{\pi_{N^*-1}} + u_{\pi_{N^*}-1} =$$

$$= \max_{\{v\} \in S_{k+1}} (u_{v_1} + \cdots + u_{v_{N^*}}) - (u_{\pi_{N^*}} - u_{\pi_{N^*} - 1}).$$

Here

$$u_{\pi_{N^*}} - u_{\pi_{N^*} - 1} \leq \max_{v, 0, 1, \ldots, k-1} (u_{v+1} - u_v).$$

Therefore, on combining the equation (33) with these two inequalities, it follows that

$$u_{k+1} - u_k \leq c_2 \log \frac{k+1}{k} + \max_{v, 0, 1, \ldots, k-1} (u_{v+1} - u_v) \quad \text{for} \quad k \geq k_0.$$

1.4. Finally put

$$v_k = u_{k+1} - u_k \quad \text{and} \quad c_4 = \max (v_0, v_1, \ldots, v_{k-1}),$$

so that $c_4$ is a further positive constant independent of $k$. Now, by (35),

$$v_k < c_2 \log \frac{k+1}{k} + \max_{v, 0, 1, \ldots, k-1} v_v \quad \text{for} \quad k \geq k_0,$$

or equivalently,

$$v_k \leq c_2 \log \frac{k+1}{k} + \max (c_4, v_{k_0}, v_{k_0+1}, \ldots, v_{k-1}) \quad \text{for} \quad k \geq k_0.$$

This inequality implies that

$$v_k \leq c_2 \log \frac{k+1}{k_0} + c_4 \quad \text{for} \quad k \geq k_0.$$  

For this assertion certainly is true if $k = k_0$. Assume then that $k > k_0$ and that the assertion has already been proved for all suffixes up to and including $k - 1$. Then

$$\max (c_4, v_{k_0}, v_{k_0+1}, \ldots, v_{k-1}) \leq c_2 \log \frac{k}{k_0} + c_4,$$

whence

$$v_k \leq c_2 \log \frac{k+1}{k} + c_2 \log \frac{k}{k_0} + c_4 = c_2 \log \frac{k+1}{k_0} + c_4.$$
This proves that the estimate (36) holds also for the suffix $k$ and therefore is always true.

On putting

$$c_5 = c_1 - c_2 \log k_0,$$

the inequality (36) shows that

$$u_{k+1} - u_k \leq c_2 \log (k + 1) + c_5 \quad \text{for} \quad k \geq k_0.$$

We apply this formula for the successive suffixes $k_0, k_0 + 1, \ldots, k - 1$, and add all the results. This leads to the estimate

$$u_k \leq u_{k_0} + c_2 \log (k!/k_0!) + c_5 (k - k_0) \quad \text{for} \quad k \geq k_0,$$

which, by (32), is equivalent to

$$|f_k| \leq e^{u_{k_0} + c_2 (k - k_0)} (k!/k_0!)^{c_5} \quad \text{for} \quad k \geq k_0.$$

In this formula, $k!$ increases more rapidly than any exponential function of $k$. We arrive then finally at the following result where we have replaced the suffix $k$ again by $h$.

**Theorem:** Let

$$f = \sum_{h=0}^{\infty} f_h z^h$$

be a formal power series with real or complex coefficients which satisfies an algebraic differential equation. Then there exist two positive constants $\gamma_1$ and $\gamma_2$ such that

$$|f_h| \leq \gamma_1 (h!)^{\gamma_2} \quad (h = 0, 1, 2, \cdots).$$

By way of example, one can easily show that if $r$ is any positive integer, then

$$f = \sum_{h=0}^{\infty} (h!)^r z^h$$

satisfies a linear differential equation, with coefficients that are polynomials in $z$. It is thus in general not possible to improve on the estimate (37).

The theorem seems to be new. In §§ 12–14, its proof makes use of an idea by a young Canberra mathematician, Mr. A. N. Stokes. For the technique of applying the algebraic differential equation to the coefficients $f_h$ I am of course greatly indebted to Popken’s doctor thesis (1935).

**References**

